Partial Fractions in Integration

When you have a rational function:

$$\frac{P(x)}{Q(x)}, \quad \deg(P) < \deg(Q),$$

you can decompose it into **simpler fractions** depending on the factors of $Q(x)$.

1. Linear Factors

If denominator has **distinct linear factors**:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

Example:

$$\frac{2x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

2. Repeated Linear Factors

If denominator has a **repeated factor**:

$$\frac{1}{(x-a)^n} \quad \Rightarrow \quad \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}$$

Example:

$$\frac{1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

3. Irreducible Quadratic Factors

If denominator has an **irreducible quadratic**:

$$\frac{1}{(x^2+bx+c)} \quad \Rightarrow \quad \frac{Ax+B}{x^2+bx+c}$$

Example:

$$\frac{3x+5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

4. Repeated Irreducible Quadratic Factors

If denominator has $(x^2 + bx + c)^n$:

$$\frac{P(x)}{(x^2 + bx + c)^n} = \frac{A_1 x + B_1}{x^2 + bx + c} + \frac{A_2 x + B_2}{(x^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(x^2 + bx + c)^n}$$

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5. General Mixed Case

For a denominator like:

$$Q(x) = (x - a)^m (x - b)^n (x^2 + px + q)^r \cdots$$

You expand using **all the above rules together**.

Integration Using Partial Fractions

After decomposition:

$$\int \frac{P(x)}{Q(x)} dx = \int \left(\frac{A}{x - a} + \frac{B}{(x - a)^2} + \cdots + \frac{Mx + N}{x^2 + px + q} \right) dx$$

- Linear terms integrate to logarithms:
  $\int \frac{dx}{x-a} = \ln |x - a|$
- Quadratic terms integrate to:
  $\int \frac{Ax+B}{x^2+px+q} dx$, which may split into $\ln$ and $\tan^{-1}$.
