Atomic Structure (

Derivation of Energy of Electron in the n-th Orbit According to Bohr

Bohr's model provides a quantized view of the hydrogen atom, where electrons orbit the nucleus in specific allowed paths without radiating energy. The energy of an electron in these orbits is also quantized. Here is the step-by-step derivation of the energy of an electron in the n-th orbit.

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1. Centripetal Force and Coulomb Force Balance

For an electron in a stable orbit around the nucleus, the centripetal force required to keep the electron in circular motion is provided by the electrostatic force of attraction between the positively charged nucleus and the negatively charged electron.

Centripetal Force:
$F_c = \frac{mv^2}{r}$

Coulomb Force:
$F_e = \frac{ke^2}{r^2}$

Here,

- m is the mass of the electron.
- v is the velocity of the electron.
- r is the radius of the electron's orbit.
- k is Coulomb's constant ($8.99 \times 10^9 \text{ Nm}^2\text{C}^{-2}$).
- e is the charge of the electron (1.602×10^{-19} C).

Setting the centripetal force equal to the Coulomb force:
$\frac{mv^2}{r} = \frac{ke^2}{r^2}$

Solving for v^2:
$v^2 = \frac{ke^2}{mr}$

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2. Quantization of Angular Momentum

Bohr proposed that the angular momentum of the electron is quantized and given by:
$L = n\hbar$

Where,

- n is the principal quantum number (1, 2, 3, ...).
- $\hbar$ is the reduced Planck's constant ($\hbar = \frac{h}{2\pi}$).

The angular momentum of the electron in circular motion is:
$L = mvr$

Equating the two expressions for angular momentum:
$mvr = n\hbar$

Solving for v:
$v = \frac{n\hbar}{mr}$

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3. Substituting for Velocity

We have two expressions for v:
$v^2 = \frac{ke^2}{mr}$
$v = \frac{n\hbar}{mr}$

Substitute v from the angular momentum quantization into the velocity squared equation:
$\left(\frac{n\hbar}{mr}\right)^2 = \frac{ke^2}{mr}$
$\frac{n^2\hbar^2}{m^2r^2} = \frac{ke^2}{mr}$

Multiply both sides by r^2 and m:
$n^2\hbar^2 = ke^2mr$

Solving for r:
$r = \frac{n^2\hbar^2}{ke^2m}$

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4. Total Energy of the Electron

The total energy (E) of the electron in the n-th orbit is the sum of its kinetic energy (K) and potential energy (U).

Kinetic Energy:
$K = \frac{1}{2}mv^2$

Using $v^2 = \frac{ke^2}{mr}$:
$K = \frac{1}{2}m \cdot \frac{ke^2}{mr} = \frac{ke^2}{2r}$

Potential Energy:
$U = -\frac{ke^2}{r}$

(The negative sign indicates that the potential energy is due to the attractive force between the nucleus and the electron.)

Total Energy:
$E = K + U = \frac{ke^2}{2r} - \frac{ke^2}{r} = -\frac{ke^2}{2r}$

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5. Substituting for r

Substitute the expression for r from earlier:
$r = \frac{n^2\hbar^2}{ke^2m}$

So,
$E = -\frac{ke^2}{2} \cdot \frac{ke^2m}{n^2\hbar^2}$
$E = -\frac{k^2e^4m}{2n^2\hbar^2}$

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6. Simplifying the Expression

Using the known constants, the energy of an electron in the n-th orbit of a hydrogen atom is given by:
$E_n = -\frac{13.6\,\text{eV}}{n^2}$

This is the energy of the electron in the n-th orbit, where:

- $13.6\,\text{eV}$ is the ground state energy of the hydrogen atom (for $n = 1$).

Summary

- The energy of an electron in the n-th orbit of a hydrogen atom is quantized and given by $E_n = -\frac{13.6\,\text{eV}}{n^2}$.
- This derivation uses the balance of centripetal and Coulomb forces, quantization of angular momentum, and the expressions for kinetic and potential energy.