

### 3. Dynamics of Rotational Motion

#### 3.1 Centre of mass

Every body is a collection of large number of tiny particles. In translatory motion of a body, every particle experiences equal displacement with time; therefore the motion of the whole body may be represented by a particle. But when the body rotates or vibrates during translatory motion, then its motion can be represented by a point on the body that moves in the same way as that of a single particle subjected to the same external forces would move. A point in the system at which whole mass of the body is supposed to be concentrated is called centre of mass of the body. Therefore, if a system contains two or more particles, its translatory motion can be described by the motion of the centre of mass of the system.

##### 3.1.1 Centre of mass of a two-particle system

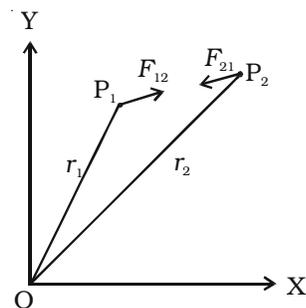
Let us consider a system consisting of two particles of masses  $m_1$  and  $m_2$ .  $P_1$  and  $P_2$  are their positions at time  $t$  and  $r_1$  and  $r_2$  are the corresponding distances from the origin O as shown in Fig. 3.1. Then the velocity and acceleration of the particles are,

$$v_1 = \frac{dr_1}{dt} \quad \dots(1)$$

$$a_1 = \frac{dv_1}{dt} \quad \dots(2)$$

$$v_2 = \frac{dr_2}{dt} \quad \dots(3)$$

$$a_2 = \frac{dv_2}{dt} \quad \dots(4)$$



The particle at  $P_1$  experiences two forces :

- (i) a force  $F_{12}$  due to the particle at  $P_2$  and  
 (ii) force  $F_{1e}$ , the external force due to some particles external to the system.

If  $F_1$  is the resultant of these two forces,

$$F_1 = F_{12} + F_{1e} \quad \dots(5)$$

Similarly, the net force  $F_2$  acting on the particle  $P_2$  is,

$$F_2 = F_{21} + F_{2e} \quad \dots(6)$$

where  $F_{21}$  is the force exerted by the particle at  $P_1$  on  $P_2$

By using Newton's second law of motion,

$$F_1 = m_1 a_1 \quad \dots(7)$$

$$\text{and } F_2 = m_2 a_2 \quad \dots(8)$$

Adding equations (7) and (8),  $m_1 a_1 + m_2 a_2 = F_1 + F_2$

Substituting  $F_1$  and  $F_2$  from (5) and (6)

$$m_1 a_1 + m_2 a_2 = F_{12} + F_{1e} + F_{21} + F_{2e}$$

By Newton's third law, the internal force  $F_{12}$  exerted by particle at  $P_2$  on the particle at  $P_1$  is equal and opposite to  $F_{21}$ , the force exerted by particle at  $P_1$  on  $P_2$ .

$$\text{(i.e) } F_{12} = -F_{21} \quad \dots(9)$$

$$\therefore F = F_{1e} + F_{2e} \quad \dots(10)$$

$$[\because m_1 a_1 + m_2 a_2 = F]$$

where  $F$  is the net external force acting on the system.

The total mass of the system is given by,

$$M = m_1 + m_2 \quad \dots(11)$$

Let the net external force  $F$  acting on the system produces an acceleration  $a_{CM}$  called the acceleration of the centre of mass of the system

By Newton's second law, for the system of two particles,

$$F = M a_{CM} \quad \dots(12)$$

$$\text{From (10) and (12), } M a_{CM} = m_1 a_1 + m_2 a_2 \quad \dots(13)$$

Let  $R_{CM}$  be the position vector of the centre of mass.

$$\therefore a_{CM} = \frac{d^2(R_{CM})}{dt^2} \quad \dots(14)$$

From (13) and (14),

$$\frac{d^2 R_{CM}}{dt^2} = \left( \frac{1}{M} \right) \left( m_1 \frac{d^2 r_1}{dt^2} + m_2 \frac{d^2 r_2}{dt^2} \right)$$

$$\begin{aligned}\frac{d^2 R_{CM}}{dt^2} &= \frac{1}{M} \left( \frac{d^2}{dt^2} (m_1 r_1 + m_2 r_2) \right) \\ \therefore R_{CM} &= \frac{1}{M} (m_1 r_1 + m_2 r_2) \\ R_{CM} &= \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} \quad \dots(15)\end{aligned}$$

This equation gives the position of the centre of mass of a system comprising two particles of masses  $m_1$  and  $m_2$

If the masses are equal ( $m_1 = m_2$ ), then the position vector of the centre of mass is,

$$R_{CM} = \frac{r_1 + r_2}{2} \quad \dots(16)$$

which means that the centre of mass lies exactly in the middle of the line joining the two masses.

### 3.1.2 Centre of mass of a body consisting of $n$ particles

For a system consisting of  $n$  particles with masses  $m_1, m_2, m_3 \dots m_n$  with position vectors  $r_1, r_2, r_3 \dots r_n$ , the total mass of the system is,

$$M = m_1 + m_2 + m_3 + \dots + m_n$$

The position vector  $R_{CM}$  of the centre of mass with respect to origin  $O$  is given by

$$R_{CM} = \frac{m_1 r_1 + m_2 r_2 + \dots + m_n r_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^n m_i r_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_i r_i}{M}$$

The  $x$  coordinate and  $y$  coordinate of the centre of mass of the system are

$$x = \frac{m_1 x_1 + m_2 x_2 + \dots + m_n x_n}{m_1 + m_2 + \dots + m_n} \quad \text{and} \quad y = \frac{m_1 y_1 + m_2 y_2 + \dots + m_n y_n}{m_1 + m_2 + \dots + m_n}$$

### Example for motion of centre of mass

Let us consider the motion of the centre of mass of the Earth and moon system (Fig 3.2). The moon moves round the Earth in a circular

orbit and the Earth moves round the Sun in an elliptical orbit. It is more correct to say that the Earth and the moon both move in circular orbits about their common centre of mass in an elliptical orbit round the Sun.

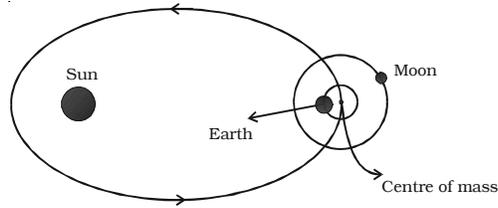


Fig 3.2 Centre of mass of Earth – moon system

For the system consisting of the Earth and the moon, their mutual gravitational attractions are the internal forces in the system and Sun's attraction on both the Earth and moon are the external forces acting on the centre of mass of the system.

### 3.1.3 Centre of gravity

A body may be considered to be made up of an indefinitely large number of particles, each of which is attracted towards the centre of the Earth by the force of gravity. These forces constitute a system of like parallel forces. The resultant of these parallel forces known as the weight of the body always acts through a point, which is fixed relative to the body, whatever be the position of the body. This fixed point is called the centre of gravity of the body.

The centre of gravity of a body is the point at which the resultant of the weights of all the particles of the body acts, whatever may be the orientation or position of the body provided that its size and shape remain unaltered.

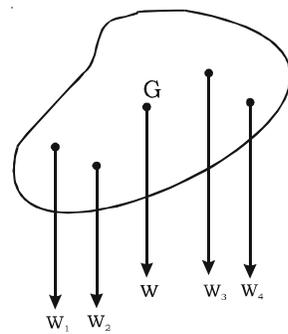


Fig . 3.3 Centre of gravity

In the Fig. 3.3,  $W_1, W_2, W_3, \dots$  are the weights of the first, second, third, ... particles in the body respectively. If  $W$  is the resultant weight of all the particles then the point at which  $W$  acts is known as the centre of gravity. The total weight of the body may be supposed to act at its centre of gravity. Since the weights of the particles constituting a body are practically proportional to their masses when the body is outside the Earth and near its surface, the centre of mass of a body practically coincides with its centre of gravity.

### 3.1.4 Equilibrium of bodies and types of equilibrium

If a marble  $M$  is placed on a curved surface of a bowl  $S$ , it rolls down and settles in equilibrium at the lowest point  $A$  (Fig. 3.4 a). This equilibrium position corresponds to minimum potential energy. If the marble is disturbed and displaced to a point  $B$ , its energy increases. When it is released, the marble rolls back to  $A$ . Thus the marble at the position  $A$  is said to be in *stable equilibrium*.

Suppose now that the bowl  $S$  is inverted and the marble is placed at its top point, at  $A$  (Fig. 3.4b). If the marble is displaced slightly to the point  $C$ , its potential energy is lowered and it tends to move further away from the equilibrium position to one of lowest energy. Thus the marble is said to be in *unstable equilibrium*.

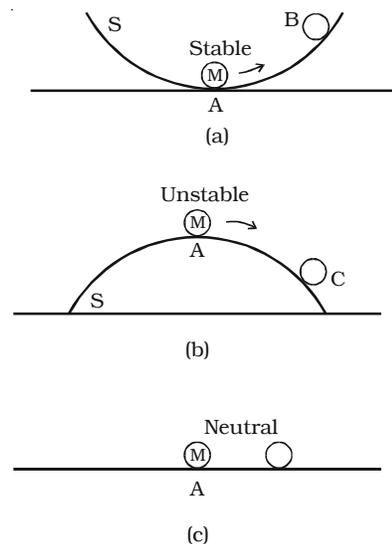


Fig.3.4 Equilibrium of rigid bodies

Suppose now that the marble is placed on a plane surface (Fig. 3.4c). If it is displaced slightly, its potential energy does not change. Here the marble is said to be in *neutral equilibrium*.

*Equilibrium is thus stable, unstable or neutral according to whether the potential energy is minimum, maximum or constant.*

We may also characterize the stability of a mechanical system by noting that when the system is disturbed from its position of equilibrium, the forces acting on the system may

(i) tend to bring back to its original position if potential energy is a *minimum*, corresponding to *stable equilibrium*.

(ii) tend to move it farther away if potential energy is *maximum*, corresponding *unstable equilibrium*.

(iii) tend to move either way if potential energy is a *constant* corresponding to *neutral equilibrium*

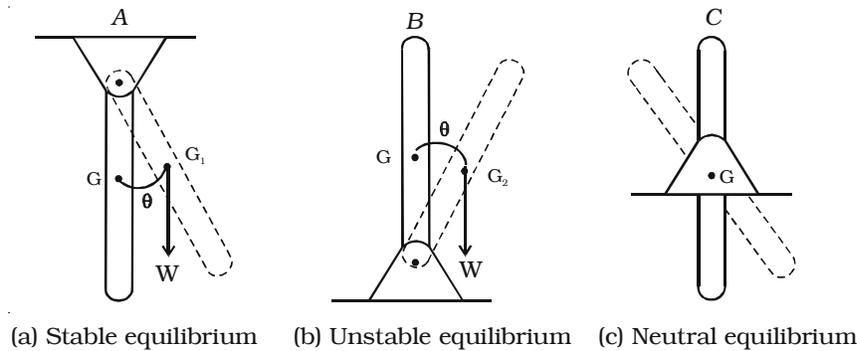


Fig 3.5 Types of equilibrium

Consider three uniform bars shown in Fig. 3.5 a,b,c. Suppose each bar is slightly displaced from its position of equilibrium and then released. For bar A, fixed at its top end, its centre of gravity  $G$  rises to  $G_1$  on being displaced, then the bar returns back to its original position on being released, so that the equilibrium is stable.

For bar B, whose fixed end is at its bottom, its centre of gravity  $G$  is lowered to  $G_2$  on being displaced, then the bar B will keep moving away from its original position on being released, and the equilibrium is said to be unstable.

For bar C, whose fixed point is about its centre of gravity, the centre of gravity remains at the same height on being displaced, the bar will remain in its new position, on being released, and the equilibrium is said to be neutral.

## 3.2 Rotational motion of rigid bodies

### 3.2.1 Rigid body

A rigid body is defined as that body which does not undergo any change in shape or volume when external forces are applied on it. When forces are applied on a rigid body, the distance between any two particles of the body will remain unchanged, however, large the forces may be.

Actually, no body is perfectly rigid. Every body can be deformed more or less by the application of the external force. The solids, in which the changes produced by external forces are negligibly small, are usually considered as rigid body.

### 3.2.2 Rotational motion

When a body rotates about a fixed axis, its motion is known as rotatory motion. A rigid body is said to have pure rotational motion, if every particle of the body moves in a circle, the centre of which lies on a straight line called the axis of rotation (Fig. 3.6). The axis of rotation may lie inside the body or even outside the body. The particles lying on the axis of rotation remains stationary.

The position of particles moving in a circular path is conveniently described in terms of a radius vector  $r$  and its angular displacement  $\theta$ . Let us consider a rigid body that rotates about a fixed axis  $XOX'$  passing through  $O$  and perpendicular to the plane of the paper as shown in Fig 3.7. Let the body rotate from the position A to the position B. The different particles at  $P_1, P_2, P_3, \dots$  in the rigid body covers unequal distances  $P_1P_1', P_2P_2',$

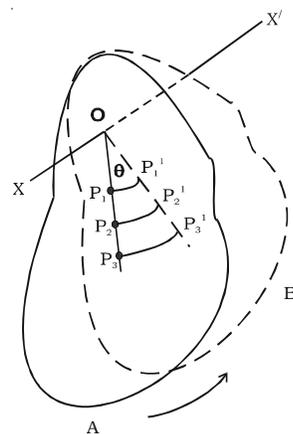


Fig 3.7 Rotational motion of a rigid body

$P_3P_3', \dots$  in the same interval of time. Thus their linear velocities are different. But in the same time interval, they all rotate through the same angle  $\theta$  and hence the angular velocity is the same for all the particles of the rigid body. Thus, in the case of rotational motion, different constituent particles have different linear velocities but all of them have the same angular velocity.

### 3.2.3 Equations of rotational motion

As in linear motion, for a body having uniform angular acceleration, we shall derive the equations of motion.

Let us consider a particle start rotating with angular velocity  $\omega_0$  and angular acceleration  $\alpha$ . At any instant  $t$ , let  $\omega$  be the angular velocity of the particle and  $\theta$  be the angular displacement produced by the particle.

Therefore change in angular velocity in time  $t = \omega - \omega_0$

But, angular acceleration =  $\frac{\text{change in angular velocity}}{\text{time taken}}$

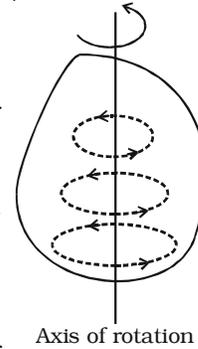


Fig 3.6 Rotational motion

$$(i.e) \quad \alpha = \frac{\omega - \omega_0}{t} \quad \dots(1)$$

$$\omega = \omega_0 + \alpha t \quad \dots(2)$$

The average angular velocity =  $\left(\frac{\omega + \omega_0}{2}\right)$

The total angular displacement  
= average angular velocity  $\times$  time taken

$$(i.e) \quad \theta = \left(\frac{\omega + \omega_0}{2}\right) t \quad \dots(3)$$

Substituting  $\omega$  from equation (2),  $\theta = \left(\frac{\omega_0 + \alpha t + \omega_0}{2}\right) t$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 \quad \dots(4)$$

From equation (1),  $t = \left(\frac{\omega - \omega_0}{\alpha}\right)$   $\dots(5)$

using equation (5) in (3),

$$\theta = \left(\frac{\omega + \omega_0}{2}\right) \left(\frac{\omega - \omega_0}{\alpha}\right) = \frac{(\omega^2 - \omega_0^2)}{2\alpha}$$

$$2\alpha \theta = \omega^2 - \omega_0^2 \quad \text{or} \quad \omega^2 = \omega_0^2 + 2\alpha \theta \quad \dots(6)$$

Equations (2), (4) and (6) are the equations of rotational motion.

### 3.3 Moment of inertia and its physical significance

According to Newton's first law of motion, a body must continue in its state of rest or of uniform motion unless it is compelled by some external agency called force. The inability of a material body to change its state of rest or of uniform motion by itself is called inertia. Inertia is the fundamental property of the matter. For a given force, the greater the mass, the higher will be the opposition for motion, or larger the inertia. Thus, in translatory motion, the mass of the body measures the coefficient of inertia.

Similarly, in rotational motion also, a body, which is free to rotate about a given axis, opposes any change desired to be produced in its state. The measure of opposition will depend on the mass of the body

and the distribution of mass about the axis of rotation. The coefficient of inertia in rotational motion is called the moment of inertia of the body about the given axis.

Moment of inertia plays the same role in rotational motion as that of mass in translatory motion. Also, to bring about a change in the state of rotation, torque has to be applied.

### 3.3.1 Rotational kinetic energy and moment of inertia of a rigid body

Consider a rigid body rotating with angular velocity  $\omega$  about an axis  $XOX'$ . Consider the particles of masses  $m_1, m_2, m_3, \dots$  situated at distances  $r_1, r_2, r_3, \dots$  respectively from the axis of rotation. The angular velocity of all the particles is same but the particles rotate with different linear velocities. Let the linear velocities of the particles be  $v_1, v_2, v_3, \dots$  respectively.

$$\text{Kinetic energy of the first particle} = \frac{1}{2} m_1 v_1^2$$

$$\text{But } v_1 = r_1 \omega$$

$\therefore$  Kinetic energy of the first particle

$$= \frac{1}{2} m_1 (r_1 \omega)^2 = \frac{1}{2} m_1 r_1^2 \omega^2$$

Similarly,

Kinetic energy of second particle

$$= \frac{1}{2} m_2 r_2^2 \omega^2$$

Kinetic energy of third particle

$$= \frac{1}{2} m_3 r_3^2 \omega^2 \text{ and so on.}$$

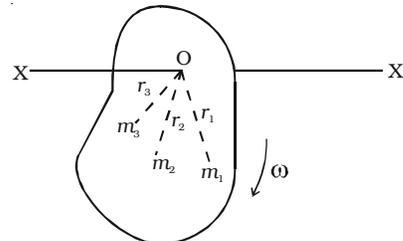


Fig. 3.8 Rotational kinetic energy and moment of inertia

The kinetic energy of the rotating rigid body is equal to the sum of the kinetic energies of all the particles.

$\therefore$  Rotational kinetic energy

$$= \frac{1}{2} (m_1 r_1^2 \omega^2 + m_2 r_2^2 \omega^2 + m_3 r_3^2 \omega^2 + \dots + m_n r_n^2 \omega^2)$$

$$= \frac{1}{2} \omega^2 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2)$$

$$(i.e) \quad E_R = \frac{1}{2} \omega^2 \left( \sum_{i=1}^n m_i r_i^2 \right) \quad \dots(1)$$

In translatory motion, kinetic energy =  $\frac{1}{2} mv^2$

Comparing with the above equation, the inertial role is played by the term  $\sum_{i=1}^n m_i r_i^2$ . This is known as moment of inertia of the rotating rigid body about the axis of rotation. Therefore the moment of inertia is

$$I = \text{mass} \times (\text{distance})^2$$

$$\text{Kinetic energy of rotation} = \frac{1}{2} \omega^2 I$$

When  $\omega = 1 \text{ rad s}^{-1}$ , rotational kinetic energy

$$= E_R = \frac{1}{2} (1)^2 I \quad (\text{or}) \quad I = 2E_R$$

*It shows that moment of inertia of a body is equal to twice the kinetic energy of a rotating body whose angular velocity is one radian per second.*

The unit for moment of inertia is  $\text{kg m}^2$  and the dimensional formula is  $\text{ML}^2$ .

### 3.3.2 Radius of gyration

The moment of inertia of the rotating rigid body is,

$$I = \sum_{i=1}^n m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2$$

If the particles of the rigid body are having same mass, then

$$m_1 = m_2 = m_3 = \dots = m \text{ (say)}$$

$\therefore$  The above equation becomes,

$$\begin{aligned} I &= m r_1^2 + m r_2^2 + m r_3^2 + \dots + m r_n^2 \\ &= m (r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2) \end{aligned}$$

$$I = nm \left[ \frac{r_1^2 + r_2^2 + r_3^2 + \dots + r_n^2}{n} \right]$$

where  $n$  is the number of particles in the rigid body.

$$\therefore I = MK^2 \quad \dots (2)$$

where  $M = nm$ , total mass of the body and  $K^2 = \frac{r_1^2 + r_2^2 + r_3^2 \dots + r_n^2}{n}$

Here  $K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 \dots + r_n^2}{n}}$  is called as the radius of gyration of the

rigid body about the axis of rotation.

*The radius of gyration is equal to the root mean square distances of the particles from the axis of rotation of the body.*

*The radius of gyration can also be defined as the perpendicular distance between the axis of rotation and the point where the whole weight of the body is to be concentrated.*

$$\text{Also from the equation (2) } K^2 = \frac{I}{M} \quad (\text{or}) \quad K = \sqrt{\frac{I}{M}}$$

### 3.3.3 Theorems of moment of inertia

#### (i) Parallel axes theorem

##### Statement

*The moment of inertia of a body about any axis is equal to the sum of its moment of inertia about a parallel axis through its centre of gravity and the product of the mass of the body and the square of the distance between the two axes.*

##### Proof

Let us consider a body having its centre of gravity at G as shown in Fig. 3.9. The axis  $XX'$  passes through the centre of gravity and is perpendicular to the plane of the body. The axis  $X_1X_1'$  passes through the point O and is parallel to the axis  $XX'$ . The distance between the two parallel axes is  $x$ .

Let the body be divided into large number of particles each of mass  $m$ . For a particle  $P$  at a distance  $r$  from O, its moment of inertia about the axis  $X_1OX_1'$  is equal to  $mr^2$ .

The moment of inertia of the whole body about the axis  $X_1X_1'$  is given by,

$$I_0 = \Sigma mr^2 \quad \dots(1)$$

From the point  $P$ , drop a perpendicular  $PA$  to the extended  $OG$  and join  $PG$ .

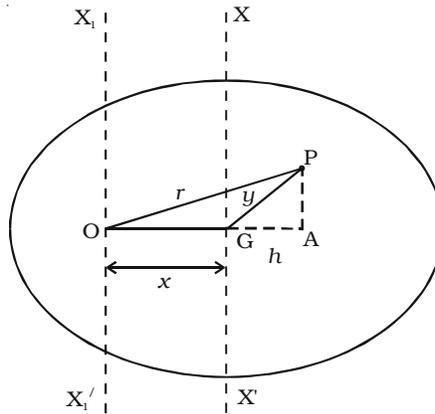


Fig .3.9 Parallel axes theorem

In the  $\triangle OPA$ ,

$$OP^2 = OA^2 + AP^2$$

$$r^2 = (x + h)^2 + AP^2$$

$$r^2 = x^2 + 2xh + h^2 + AP^2 \quad \dots(2)$$

But from  $\triangle GPA$ ,

$$GP^2 = GA^2 + AP^2$$

$$y^2 = h^2 + AP^2 \quad \dots(3)$$

Substituting equation (3) in (2),

$$r^2 = x^2 + 2xh + y^2 \quad \dots(4)$$

Substituting equation (4) in (1),

$$\begin{aligned} I_o &= \Sigma m (x^2 + 2xh + y^2) \\ &= \Sigma mx^2 + \Sigma 2mxh + \Sigma my^2 \\ &= Mx^2 + My^2 + 2x\Sigma mh \end{aligned} \quad \dots(5)$$

Here  $My^2 = I_G$  is the moment of inertia of the body about the line passing through the centre of gravity. The sum of the turning moments of

all the particles about the centre of gravity is zero, since the body is balanced about the centre of gravity G.

$$\Sigma (mg) (h) = 0 \quad (\text{or}) \quad \Sigma mh = 0 \quad [\text{since } g \text{ is a constant}] \quad \dots(6)$$

$$\therefore \text{equation (5) becomes, } I_o = Mx^2 + I_G \quad \dots(7)$$

Thus the parallel axes theorem is proved.

### (ii) Perpendicular axes theorem

#### Statement

The moment of inertia of a plane lamina about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two mutually perpendicular axes in the plane of the lamina such that the three mutually perpendicular axes have a common point of intersection.

#### Proof

Consider a plane lamina having the axes OX and OY in the plane of the lamina as shown Fig. 3.10. The axis OZ passes through O and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles, each of mass  $m$ . A particle at P at a distance  $r$  from O has coordinates  $(x,y)$ .

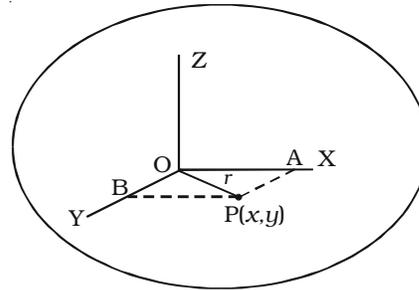


Fig 3.10 Perpendicular axes theorem

$$\therefore r^2 = x^2 + y^2 \quad \dots(1)$$

The moment of inertia of the particle P about the axis OZ =  $m r^2$ .

The moment of inertia of the whole lamina about the axis OZ is

$$I_z = \Sigma mr^2 \quad \dots(2)$$

The moment of inertia of the whole lamina about the axis OX is

$$I_x = \Sigma my^2 \quad \dots(3)$$

Similarly,  $I_y = \Sigma mx^2 \quad \dots(4)$

From eqn. (2),  $I_z = \Sigma mr^2 = \Sigma m(x^2 + y^2)$

$$I_z = \Sigma mx^2 + \Sigma my^2 = I_y + I_x$$

$$\therefore I_z = I_x + I_y$$

which proves the perpendicular axes theorem.

**Table 3.1 Moment of Inertia of different bodies**

(Proof is given in the annexure)

Body	Axis of Rotation	Moment of Inertia
Thin Uniform Rod	Axis passing through its centre of gravity and perpendicular to its length	$\frac{Ml^2}{12}$ M - mass l - length
	Axis passing through the end and perpendicular to its length.	$\frac{Ml^2}{3}$ M - mass l - length
Thin Circular Ring	Axis passing through its centre and perpendicular to its plane.	$MR^2$ M - mass R - radius
	Axis passing through its diameter	$\frac{1}{2}MR^2$ M - mass R - radius
	Axis passing through a tangent	$\frac{3}{2}MR^2$ M - mass R - radius
Circular Disc	Axis passing through its centre and perpendicular to its plane.	$\frac{1}{2}MR^2$ M - mass R - radius
	Axis passing through its diameter	$\frac{1}{4}MR^2$ M - mass R - radius
	Axis passing through a tangent	$\frac{5}{4}MR^2$ M - mass R - radius
Solid Sphere	Axis passing through its diameter	$\frac{2}{5}MR^2$ M - mass R - radius
	Axis passing through a tangent	$\frac{7}{5}MR^2$ M - mass R - radius
Solid Cylinder	Its own axis	$\frac{1}{2}MR^2$ M - mass R - radius
	Axis passing through its centre and perpendicular to its length	$M \left( \frac{R^2}{4} + \frac{l^2}{12} \right)$ M - mass R - radius l - length

### 3.4 Moment of a force

A force can rotate a nut when applied by a wrench or it can open a door while the door rotates on its hinges (i.e) in addition to the tendency to move a body in the direction of the application of a force, a force also tends to rotate the body about any axis which does not intersect the line of action of the force and also not parallel to it. This tendency of rotation is called turning effect of a force or moment of the force about the given axis. *The magnitude of the moment of force  $F$  about a point is defined as the product of the magnitude of force and the perpendicular distance of the point from the line of action of the force.*

Let us consider a force  $F$  acting at the point  $P$  on the body as shown in Fig. 3.11. Then, the moment of the force  $F$  about the point  $O = \text{Magnitude of the force} \times \text{perpendicular distance between the direction of the force and the point about which moment is to be determined} = F \times OA$ .

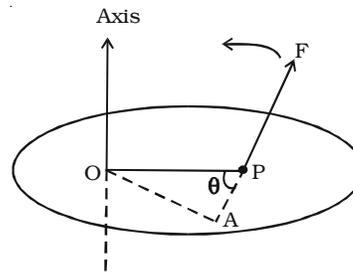


Fig 3.11 Moment of a force

*If the force acting on a body rotates the body in anticlockwise direction with respect to  $O$  then the moment is called anticlockwise moment. On the other hand, if the force rotates the body in clockwise direction then the moment is said to be clockwise moment. The unit of moment of the force is  $N\ m$  and its dimensional formula is  $M\ L^2\ T^{-2}$ .*

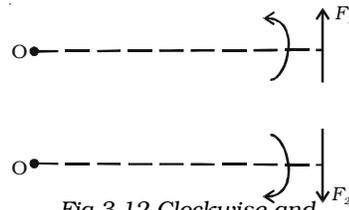


Fig 3.12 Clockwise and anticlockwise moments

*As a matter of convention, an anticlockwise moment is taken as positive and a clockwise moment as negative. While adding moments, the direction of each moment should be taken into account.*

In terms of vector product, the moment of a force is expressed as,

$$\vec{m} = \vec{r} \times \vec{F}$$

where  $\vec{r}$  is the position vector with respect to  $O$ . The direction of  $\vec{m}$  is perpendicular to the plane containing  $\vec{r}$  and  $\vec{F}$ .

### 3.5 Couple and moment of the couple (Torque)

There are many examples in practice where two forces, acting together, exert a moment, or turning effect on some object. As a very simple case, suppose two strings are tied to a wheel at the points  $X$  and  $Y$ , and *two equal and opposite forces*,  $F$ , are exerted tangentially to the wheels (Fig. 3.13). If the wheel is pivoted at its centre  $O$  it begins to rotate about  $O$  in an anticlockwise direction.

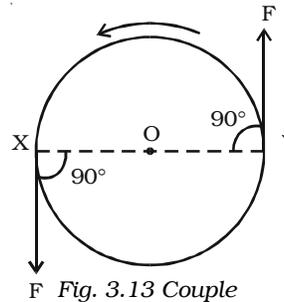


Fig. 3.13 Couple

*Two equal and opposite forces whose lines of action do not coincide are said to constitute a couple* in mechanics. The two forces always have a *turning effect*, or *moment*, called a *torque*. The perpendicular distance between the lines of action of two forces, which constitute the couple, is called the *arm of the couple*.

*The product of the forces forming the couple and the arm of the couple is called the moment of the couple or torque.*

Torque = one of the forces  $\times$  perpendicular distance between the forces

The torque in rotational motion plays the same role as the force in translational motion. A quantity that is a measure of this rotational effect produced by the force is called torque.

In vector notation,  $\vec{\tau} = \vec{r} \times \vec{F}$

The torque is maximum when  $\theta = 90^\circ$  (i.e) when the applied force is at right angles to  $\vec{r}$ .

#### Examples of couple are

1. Forces applied to the handle of a screw press,
2. Opening or closing a water tap.
3. Turning the cap of a pen.
4. Steering a car.

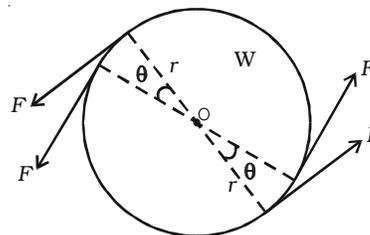


Fig.3.14 Work done by a couple

#### Work done by a couple

Suppose two equal and opposite forces  $F$  act tangentially to a wheel  $W$ , and rotate it through an angle  $\theta$  (Fig. 3.14).

Then the work done by each force = Force  $\times$  distance =  $F \times r \theta$   
(since  $r \theta$  is the distance moved by a point on the rim)

Total work done  $W = F r \theta + F r \theta = 2F r \theta$

but torque  $\tau = F \times 2r = 2F r$

$\therefore$  work done by the couple,  $W = \tau \theta$

### 3.6 Angular momentum of a particle

The angular momentum in a rotational motion is similar to the linear momentum in translatory motion. The linear momentum of a particle moving along a straight line is the product of its mass and linear velocity (i.e)  $p = mv$ . The angular momentum of a particle is defined as the moment of linear momentum of the particle.

Let us consider a particle of mass  $m$  moving in the  $XY$  plane with a velocity  $v$  and linear momentum  $\vec{p} = m\vec{v}$  at a distance  $r$  from the origin (Fig. 3.15).

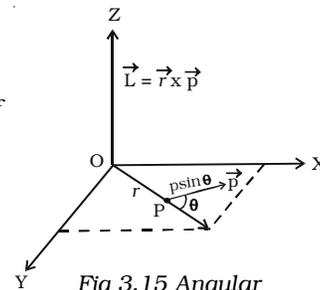


Fig 3.15 Angular momentum of a particle

The angular momentum  $L$  of the particle about an axis passing through  $O$  perpendicular to  $XY$  plane is defined as the cross product of  $\vec{r}$  and  $\vec{p}$ .

$$(i.e) \vec{L} = \vec{r} \times \vec{p}$$

Its magnitude is given by  $L = r p \sin \theta$

where  $\theta$  is the angle between  $\vec{r}$  and  $\vec{p}$  and  $L$  is along a direction perpendicular to the plane containing  $\vec{r}$  and  $\vec{p}$ .

The unit of angular momentum is  $\text{kg m}^2 \text{s}^{-1}$  and its dimensional formula is,  $M L^2 T^{-1}$ .

#### 3.6.1 Angular momentum of a rigid body

Let us consider a system of  $n$  particles of masses  $m_1, m_2, \dots, m_n$  situated at distances  $r_1, r_2, \dots, r_n$  respectively from the axis of rotation (Fig. 3.16). Let  $v_1, v_2, v_3, \dots$  be the linear velocities of the particles respectively, then linear momentum of first particle =  $m_1 v_1$ .

Since  $v_1 = r_1 \omega$  the linear momentum of first particle =  $m_1(r_1 \omega)$

The moment of linear momentum of first particle

= linear momentum  $\times$   
perpendicular distance

$$= (m_1 r_1 \omega) \times r_1$$

angular momentum of first

$$\text{particle} = m_1 r_1^2 \omega$$

Similarly,

$$\text{angular momentum of second particle} = m_2 r_2^2 \omega$$

$$\text{angular momentum of third particle} = m_3 r_3^2 \omega \text{ and so on.}$$

The sum of the moment of the linear momenta of all the particles of a rotating rigid body taken together about the axis of rotation is known as angular momentum of the rigid body.

$\therefore$  Angular momentum of the rotating rigid body = sum of the angular momenta of all the particles.

$$\text{(i.e.) } L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + \dots + m_n r_n^2 \omega$$

$$L = \omega [m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots + m_n r_n^2]$$

$$= \omega \left[ \sum_{i=1}^n m_i r_i^2 \right]$$

$$\therefore L = \omega I$$

where  $I = \sum_{i=1}^n m_i r_i^2$  = moment of inertia of the rotating rigid body about the axis of rotation.

### 3.7 Relation between torque and angular acceleration

Let us consider a rigid body rotating about a fixed axis XOX' with angular velocity  $\omega$  (Fig. 3.17).

The force acting on a particle of mass  $m_1$  situated at A, at a distance  $r_1$ , from the axis of rotation = mass  $\times$  acceleration

$$= m_1 \times \frac{d}{dt}(r_1 \omega)$$

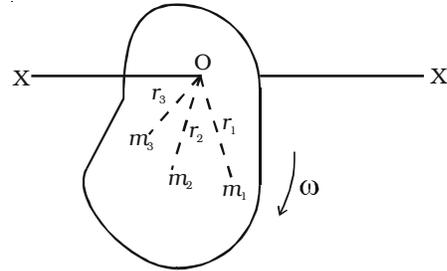


Fig 3.16 Angular momentum of a rigid body

$$= m_1 r_1 \frac{d\omega}{dt}$$

$$= m_1 r_1 \frac{d^2\theta}{dt^2}$$

The moment of this force about the axis of rotation

= Force  $\times$  perpendicular distance

$$= m_1 r_1 \frac{d^2\theta}{dt^2} \times r_1$$

Therefore, the total moment of all

the forces acting on all the particles

$$= m_1 r_1^2 \frac{d^2\theta}{dt^2} + m_2 r_2^2 \frac{d^2\theta}{dt^2} + \dots$$

$$\text{(i.e) torque} = \sum_{i=1}^n m_i r_i^2 \times \frac{d^2\theta}{dt^2}$$

$$\text{or } \tau = I\alpha$$

where  $\sum_{i=1}^n m_i r_i^2 =$  moment of inertia  $I$  of the rigid body and  $\alpha = \frac{d^2\theta}{dt^2}$  angular acceleration.

### 3.7.1 Relation between torque and angular momentum

The angular momentum of a rotating rigid body is,  $L = I\omega$

Differentiating the above equation with respect to time,

$$\frac{dL}{dt} = I \left( \frac{d\omega}{dt} \right) = I\alpha$$

where  $\alpha = \frac{d\omega}{dt}$  angular acceleration of the body.

But torque  $\tau = I\alpha$

Therefore, torque  $\tau = \frac{dL}{dt}$

Thus the rate of change of angular momentum of a body is equal to the external torque acting upon the body.

### 3.8 Conservation of angular momentum

The angular momentum of a rotating rigid body is,  $L = I\omega$

The torque acting on a rigid body is,  $\tau = \frac{dL}{dt}$

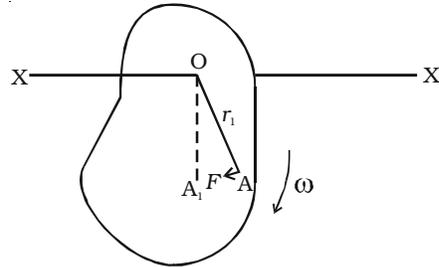


Fig 3.17 Relation between torque and angular acceleration

When no external torque acts on the system,  $\tau = \frac{dL}{dt} = 0$

(i.e)  $L = I\omega = \text{constant}$

Total angular momentum of the body = constant

(i.e.) when no external torque acts on the body, the net angular momentum of a rotating rigid body remains constant. This is known as law of conservation of angular momentum.

### **Illustration of conservation of angular momentum**

From the law of conservation of angular momentum,  $I\omega = \text{constant}$

(ie)  $\omega \propto \frac{1}{I}$ , the angular velocity of rotation is inversely proportional

to the moment of inertia of the system.

Following are the examples for law of conservation of angular momentum.

1. A diver jumping from springboard sometimes exhibits somersaults in air before reaching the water surface, because the diver curls his body to decrease the moment of inertia and increase angular velocity. When he

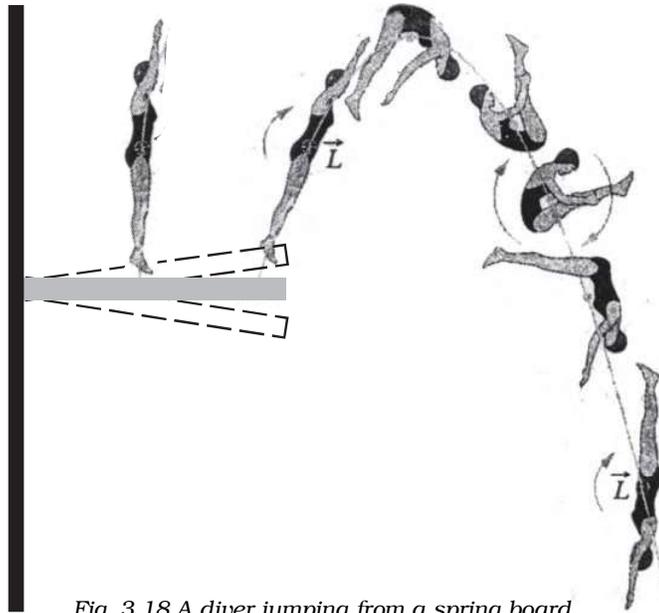


Fig. 3.18 A diver jumping from a spring board

is about to reach the water surface, he again outstretches his limbs. This again increases moment of inertia and decreases the angular velocity. Hence, the diver enters the water surface with a gentle speed.

2. A ballet dancer can increase her angular velocity by folding her arms, as this decreases the moment of inertia.

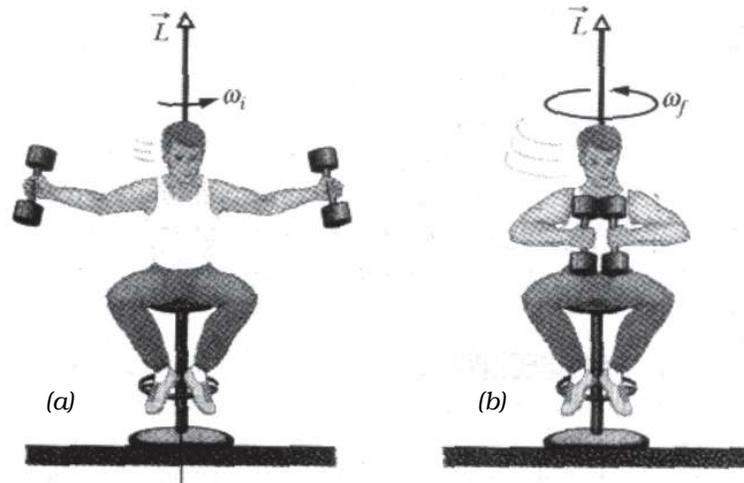


Fig 3.19 A person rotating on a turn table

3. Fig. 3.19a shows a person sitting on a turntable holding a pair of heavy dumbbells one in each hand with arms outstretched. The table is rotating with a certain angular velocity. The person suddenly pushes the weight towards his chest as shown Fig. 3.19b, the speed of rotation is found to increase considerably.

4. The angular velocity of a planet in its orbit round the sun increases when it is nearer to the Sun, as the moment of inertia of the planet about the Sun decreases.

### Solved Problems

- 3.1 A system consisting of two masses connected by a massless rod lies along the X-axis. A 0.4 kg mass is at a distance  $x = 2$  m while a 0.6 kg mass is at  $x = 7$  m. Find the x coordinate of the centre of mass.

**Data :**  $m_1 = 0.4$  kg ;  $m_2 = 0.6$  kg ;  $x_1 = 2$  m ;  $x_2 = 7$  m ;  $x = ?$

**Solution :**  $x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{(0.4 \times 2) + (0.6 \times 7)}{(0.4 + 0.6)} = 5$  m

- 3.2 Locate the centre of mass of a system of bodies of masses  $m_1 = 1$  kg,  $m_2 = 2$  kg and  $m_3 = 3$  kg situated at the corners of an equilateral triangle of side 1 m.

**Data :**  $m_1 = 1$  kg ;  $m_2 = 2$  kg ;  $m_3 = 3$  kg ;

The coordinates of A = (0,0)

The coordinates of B = (1,0)

Centre of mass of the system =?

**Solution :** Consider an equilateral triangle of side 1m as shown in Fig. Take X and Y axes as shown in figure.

To find the coordinate of C:

For an equilateral triangle ,  
 $\angle CAB = 60^\circ$

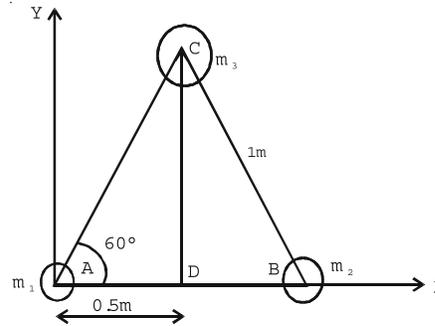
Consider the triangle ADC,

$$\sin \theta = \frac{CD}{CA} \quad (\text{or}) \quad CD =$$

$$(CA) \sin \theta = 1 \times \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Therefore from the figure, the coordinate of C are,  $(0.5, \frac{\sqrt{3}}{2})$

$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$



$$x = \frac{(1 \times 0) + (2 \times 1) + (3 \times 0.5)}{(1 + 2 + 3)} = \frac{3.5}{6} m$$

$$y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$y = \frac{(1 \times 0) + (2 \times 0) + \left(3 \times \frac{\sqrt{3}}{2}\right)}{6} = \frac{\sqrt{3}}{4} m$$

3.3 A circular disc of mass  $m$  and radius  $r$  is set rolling on a table.

If  $\omega$  is its angular velocity, show that its total energy  $E = \frac{3}{4} mr^2 \omega^2$ .

**Solution :** The total energy of the disc = Rotational KE + linear KE

$$\therefore E = \frac{1}{2} I \omega^2 + \frac{1}{2} m v^2 \quad \dots(1)$$

$$\text{But } I = \frac{1}{2} m r^2 \text{ and } v = r \omega \quad \dots(2)$$

Substituting eqn. (2) in eqn. (1),

$$\begin{aligned} E &= \frac{1}{2} \times \frac{1}{2} (m r^2) (\omega^2) + \frac{1}{2} m (r \omega)^2 = \frac{1}{4} m r^2 \omega^2 + \frac{1}{2} m r^2 \omega^2 \\ &= \frac{3}{4} m r^2 \omega^2 \end{aligned}$$

3.4 A thin metal ring of diameter 0.6m and mass 1kg starts from rest and rolls down on an inclined plane. Its linear velocity on reaching the foot of the plane is  $5 \text{ m s}^{-1}$ , calculate (i) the moment of inertia of the ring and (ii) the kinetic energy of rotation at that instant.

**Data :**  $R = 0.3 \text{ m}$  ;  $M = 1 \text{ kg}$  ;  $v = 5 \text{ m s}^{-1}$  ;  $I = ?$  K.E. = ?

**Solution :**  $I = MR^2 = 1 \times (0.3)^2 = 0.09 \text{ kg m}^2$

$$\text{K.E.} = \frac{1}{2} I \omega^2$$

$$v = r \omega ; \therefore \omega = \frac{v}{r} ; \text{K.E.} = \frac{1}{2} \times 0.09 \times \left(\frac{5}{0.3}\right)^2 = 12.5 \text{ J}$$

- 3.5 A solid cylinder of mass 200 kg rotates about its axis with angular speed  $100 \text{ s}^{-1}$ . The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of the angular momentum of the cylinder about its axis?

**Data :**  $M = 200 \text{ kg}$  ;  $\omega = 100 \text{ s}^{-1}$  ;  $R = 0.25 \text{ metre}$  ;  
 $E_R = ?$  ;  $L = ?$

**Solution :**  $I = \frac{MR^2}{2} = \frac{200 \times (0.25)^2}{2} = 6.25 \text{ kg m}^2$

$$\begin{aligned} \text{K.E.} &= \frac{1}{2} I \omega^2 \\ &= \frac{1}{2} \times 6.25 \times (100)^2 \end{aligned}$$

$$E_R = 3.125 \times 10^4 \text{ J}$$

$$L = I\omega = 6.25 \times 100 = 625 \text{ kg m}^2 \text{ s}^{-1}$$

- 3.6 Calculate the radius of gyration of a rod of mass 100 g and length 100 cm about an axis passing through its centre of gravity and perpendicular to its length.

**Data :**  $M = 100 \text{ g} = 0.1 \text{ kg}$  ;  $l = 100 \text{ cm} = 1 \text{ m}$

$K = ?$

**Solution :** The moment of inertia of the rod about an axis passing through its centre of gravity and perpendicular to the length =  $I =$

$$MK^2 = \frac{ML^2}{12} \quad (\text{or}) \quad K^2 = \frac{L^2}{12} \quad (\text{or}) \quad K = \frac{L}{\sqrt{12}} = \frac{1}{\sqrt{12}} = 0.2886 \text{ m.}$$

- 3.7 A circular disc of mass 100 g and radius 10 cm is making 2 revolutions per second about an axis passing through its centre and perpendicular to its plane. Calculate its kinetic energy.

**Data :**  $M = 100 \text{ g} = 0.1 \text{ kg}$  ;  $R = 10 \text{ cm} = 0.1 \text{ m}$  ;  $n = 2$

**Solution :**  $\omega = \text{angular velocity} = 2\pi n = 2\pi \times 2 = 4\pi \text{ rad / s}$

$$\begin{aligned} \text{Kinetic energy of rotation} &= \frac{1}{2} I\omega^2 \\ &= \frac{1}{2} \times \frac{1}{2} \times MR^2 \omega^2 = \frac{1}{2} \times \frac{1}{2} (0.1) \times (0.1)^2 \times (4\pi)^2 \\ &= 3.947 \times 10^{-2} \text{ J} \end{aligned}$$

- 3.8 Starting from rest, the flywheel of a motor attains an angular velocity 100 rad/s from rest in 10 s. Calculate (i) angular acceleration and (ii) angular displacement in 10 seconds.

**Data :**  $\omega_0 = 0$  ;  $\omega = 100 \text{ rad s}^{-1}$      $t = 10 \text{ s}$      $\alpha = ?$

**Solution :** From equations of rotational dynamics,

$$\omega = \omega_0 + \alpha t$$

$$\text{(or) } \alpha = \frac{\omega - \omega_0}{t} = \frac{100 - 0}{10} = 10 \text{ rad s}^{-2}$$

$$\text{Angular displacement } \theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$= 0 + \frac{1}{2} \times 10 \times 10^2 = 500 \text{ rad}$$

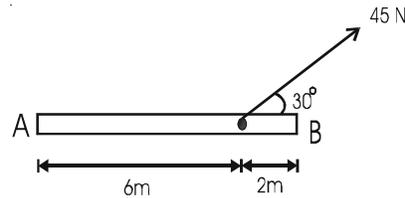
- 3.9 A disc of radius 5 cm has moment of inertia of 0.02 kg m<sup>2</sup>. A force of 20 N is applied tangentially to the surface of the disc. Find the angular acceleration produced.

**Data :**  $I = 0.02 \text{ kg m}^2$  ;  $r = 5 \text{ cm} = 5 \times 10^{-2} \text{ m}$  ;  $F = 20 \text{ N}$  ;  $\tau = ?$

**Solution :** Torque =  $\tau = F \times r = 20 \times 2 \times 5 \times 10^{-2} = 2 \text{ N m}$

angular acceleration =  $\alpha = \frac{\tau}{I} = \frac{2}{0.02} = 100 \text{ rad /s}^2$

- 3.10 From the figure, find the moment of the force 45 N about A?

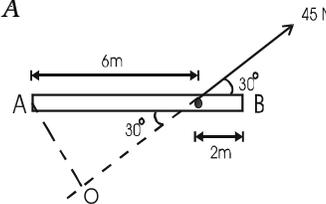


**Data :** Force  $F = 45 \text{ N}$  ; Moment of the force about A = ?

**Solution :** Moment of the force about A

$$= \text{Force} \times \text{perpendicular distance} = F \times AO$$

$$= 45 \times 6 \sin 30 = 135 \text{ N m}$$



### Self evaluation

*(The questions and problems given in this self evaluation are only samples. In the same way any question and problem could be framed from the text matter. Students must be prepared to answer any question and problem from the text matter, not only from the self evaluation.)*

- 3.1 The angular speed of minute arm in a watch is :
- (a)  $\pi/21600 \text{ rad s}^{-1}$                       (b)  $\pi/12 \text{ rad s}^{-1}$   
 (c)  $\pi/3600 \text{ rad s}^{-1}$                       (d)  $\pi/1800 \text{ rad s}^{-1}$
- 3.2 The moment of inertia of a body comes into play
- (a) in linear motion                      (b) in rotational motion  
 (c) in projectile motion                      (d) in periodic motion
- 3.3 Rotational analogue of mass in linear motion is
- (a) Weight                      (b) Moment of inertia  
 (c) Torque                      (d) Angular momentum
- 3.4 The moment of inertia of a body does not depend on
- (a) the angular velocity of the body  
 (b) the mass of the body  
 (c) the axis of rotation of the body  
 (d) the distribution of mass in the body
- 3.5 A ring of radius  $r$  and mass  $m$  rotates about an axis passing through its centre and perpendicular to its plane with angular velocity  $\omega$ . Its kinetic energy is
- (a)  $m r \omega^2$                       (b)  $\frac{1}{2} m r \omega^2$                       (c)  $I \omega^2$                       (d)  $\frac{1}{2} I \omega^2$
- 3.6 The moment of inertia of a disc having mass  $M$  and radius  $R$ , about an axis passing through its centre and perpendicular to its plane is
- (a)  $\frac{1}{2} M R^2$                       (b)  $M R^2$                       (c)  $\frac{1}{4} M R^2$                       (d)  $\frac{5}{4} M R^2$
- 3.7 Angular momentum is the vector product of
- (a) linear momentum and radius vector  
 (b) moment of inertia and angular velocity  
 (c) linear momentum and angular velocity  
 (d) linear velocity and radius vector

- 3.8 The rate of change of angular momentum is equal to  
(a) Force (b) Angular acceleration  
(c) Torque (d) Moment of Inertia
- 3.9 Angular momentum of the body is conserved  
(a) always  
(b) never  
(c) in the absence of external torque  
(d) in the presence of external torque
- 3.10 A man is sitting on a rotating stool with his arms outstretched. Suddenly he folds his arm. The angular velocity  
(a) decreases (b) increases  
(c) becomes zero (d) remains constant
- 3.11 An athlete diving off a high springboard can perform a variety of exercises in the air before entering the water below. Which one of the following parameters will remain constant during the fall. The athlete's  
(a) linear momentum (b) moment of inertia  
(c) kinetic energy (d) angular momentum
- 3.12 Obtain an expression for position of centre of mass of two particle system.
- 3.13 Explain the motion of centre of mass of a system with an example.
- 3.14 What are the different types of equilibrium?
- 3.15 Derive the equations of rotational motion.
- 3.16 Compare linear motion with rotational motion.
- 3.17 Explain the physical significance of moment of inertia.
- 3.18 Show that the moment of inertia of a rigid body is twice the kinetic energy of rotation.
- 3.19 State and prove parallel axes theorem and perpendicular axes theorem.
- 3.20 Obtain the expressions for moment of inertia of a ring (i) about an axis passing through its centre and perpendicular to its plane. (ii) about its diameter and (iii) about a tangent.