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Volume of a Solid Obtained by Revolving an Area About an Axis

The method of finding the volume of a solid formed by revolving a region around an axis is a fundamental application of integration. This is commonly done using the **disk method** or the **shell method**, depending on the axis of rotation and the geometry of the problem.

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1. Disk Method

When a region is revolved about an axis (e.g., x-axis or y-axis), the volume of the resulting solid can be calculated by summing up the volumes of infinitesimally thin disks.

Formula

For a region bounded by $y = f(x)$ from $x = a$ to $x = b$ revolved around the x-axis:

$$V = \pi \int_a^b [f(x)]^2 \, dx$$

Here:

- $f(x)$ represents the radius of the disk.
- dx represents the thickness of the disk.

For rotation about the y-axis, with $x = g(y)$ and $y \in [c, d]$:

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

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2. Washer Method

If the solid has a hollow region (like a doughnut), we use the washer method. The volume is the difference between the volumes of the outer and inner solids.

Formula

For a region bounded by $y = f(x)$ (outer radius) and $y = g(x)$ (inner radius), rotated about the x-axis:

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx$$

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3. Shell Method

The shell method calculates the volume by summing up cylindrical shells formed by revolving a region about an axis.

Formula

For a region bounded by $y = f(x)$, $x \in [a, b]$, revolved about the y-axis:

$$V = 2\pi \int_a^b x f(x)\, dx$$

Here:

- x is the radius of the shell.
- $f(x)$ is the height of the shell.

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4. Examples

Example 1: Disk Method

Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$, $x = 0$, and $x = 4$ about the x-axis.

1. **Set up the integral:** The radius of the disk is $y = \sqrt{x}$, so the volume is:

$$V = \pi \int_0^4 (\sqrt{x})^2\, dx$$

2. **Simplify and compute:**

$$V = \pi \int_0^4 x\, dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \left(\frac{16}{2} - 0\right) = 8\pi$$

3. **Result:**

$$V = 8\pi \text{ cubic units.}$$

Example 2: Washer Method

Find the volume of the solid obtained by revolving the region between $y = x^2$ and $y = 4$ about the x-axis.

1. **Set up the integral:** The outer radius is $R(x) = 2$, and the inner radius is $r(x) = x^2$:

$$V = \pi \int_0^2 \left([2]^2 - [x^2]^2\right) dx$$

2. **Simplify:**

$$V = \pi \int_0^2 \left(4 - x^4\right) dx$$

3. **Integrate:**

$$V = \pi \left[4x - \frac{x^5}{5}\right]_0^2 = \pi\left(8 - \frac{32}{5} - 0\right) = \pi\left(\frac{40}{5} - \frac{32}{5}\right) = \frac{8\pi}{5}$$

4. **Result:**

$$V = \frac{8\pi}{5} \text{ cubic units.}$$

Example 3: Shell Method

Find the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $x = 2$ about the y -axis.

1. **Set up the integral:** Using the shell method, the height of the shell is $f(x) = x^2$, and the radius is x:

$$V = 2\pi \int_0^2 x \cdot x^2 \, dx = 2\pi \int_0^2 x^3 \, dx$$

2. **Integrate:**

$$V = 2\pi \left[\frac{x^4}{4}\right]_0^2 = 2\pi\left(\frac{16}{4} - 0\right) = 2\pi \cdot 4 = 8\pi$$

3. **Result:**

$$V = 8\pi \text{ cubic units.}$$

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5. Applications

1. **Engineering**:
    - Calculating the volume of tanks, pipes, and other rotational structures.
2. **Physics**:
    - Finding moments of inertia and mass distribution in rotational bodies.
3. **Architecture**:
    - Designing domes, arches, and curved surfaces with precise volumes.

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6. Key Points

- Use the **disk method** when the solid has no hollow region and is revolved about an axis.
- Use the **washer method** when the solid has a hollow core.
- Use the **shell method** when the region is easier to slice parallel to the axis of rotation.