Saitechinfo NEET-JEE Academy

Reduction Formulae: Lecture Notes

Reduction formulae are recursive relationships that simplify the evaluation of integrals involving powers of functions like trigonometric, exponential, or polynomial terms. These formulae help reduce higher-order integrals into simpler ones by expressing them in terms of integrals of lower powers.

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1. Concept of Reduction Formula

A **reduction formula** relates an integral of a function raised to a power n to an integral of the same function raised to a lower power $n-1$ or $n-2$.

General Form

$$I_n = \int f(x)^n \, g(x) \, dx$$

is expressed as:

$$I_n = F(n, I_{n-1}, I_{n-2}, \dots)$$

where F is a function of n and simpler integrals like $I_{n-1}, I_{n-2}, \dots$.

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2. Derivation of Reduction Formulae

Case 1: Powers of Sine or Cosine

For $I_n = \int \sin^n(x) \, dx$, use the identity:

$$\sin^n(x) = \sin^{n-2}(x) \cdot \sin^2(x)$$

and the Pythagorean identity:

$$\sin^2(x) = 1 - \cos^2(x).$$

Reduction Formula for $\sin^n(x)$:

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{\cos(x) \sin^{n-1}(x)}{n}.$$

Case 2: Powers of Exponential Functions

For $I_n = \int x^n e^x \, dx$, use **integration by parts**:

$$I_n = x^n e^x - n \int x^{n-1} e^x \, dx.$$

Reduction Formula for $x^n e^x$:

$$I_n = x^n e^x - nI_{n-1}.$$

Case 3: Powers of Trigonometric Functions

For $I_n = \int \sin^n(x)\cos^m(x)\,dx$, use **substitution** and **trigonometric identities**.

If m is odd, substitute $u = \sin(x)$, $du = \cos(x)\,dx$.

If n is odd, substitute $u = \cos(x)$, $du = -\sin(x)\,dx$.

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3. Examples of Reduction Formulae

Example 1: Reduction Formula for $\int \sin^n(x)\,dx$

1. Let $I_n = \int \sin^n(x)\,dx$.
2. Use integration by parts:

$$I_n = -\cos(x)\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,dx.$$

3. Express as a reduction formula:

$$I_n = -\frac{\cos(x)\sin^{n-1}(x)}{n} + \frac{n-1}{n}I_{n-2}.$$

Example 2: Reduction Formula for $\int x^n e^x\,dx$

1. Let $I_n = \int x^n e^x\,dx$.
2. Use integration by parts:

$$I_n = x^n e^x - n\int x^{n-1}e^x\,dx.$$

3. Express as:

$$I_n = x^n e^x - nI_{n-1}.$$

Example 3: Reduction Formula for $\int \tan^n(x)\,dx$

1. Let $I_n = \int \tan^n(x)\,dx$.
2. Use the identity:

$$\tan^n(x) = \tan^{n-2}(x)\sec^2(x) - \int \tan^{n-2}(x)\,dx.$$

3. Reduction formula:

$$I_n = \frac{\tan^{n-1}(x)}{n-1} - \frac{1}{n-1}I_{n-2}.$$

4. Applications

Example 1: Evaluate $\int \sin^4(x)\,dx$

1. Use the reduction formula for $\sin^n(x)$:

$$I_n = -\frac{\cos(x)\sin^{n-1}(x)}{n} + \frac{n-1}{n} I_{n-2}.$$

2. For $n = 4$:

$$I_4 = -\frac{\cos(x)\sin^3(x)}{4} + \frac{3}{4} I_2.$$

3. Use $I_2 = \int \sin^2(x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4}$.
4. Substitute I_2 back to compute I_4.

Example 2: Evaluate $\int x^3 e^x\,dx$

1. Use the reduction formula:

$$I_n = x^n e^x - nI_{n-1}.$$

2. For $n = 3$:

$$I_3 = x^3 e^x - 3I_2.$$

3. Compute $I_2 = x^2 e^x - 2I_1$, and $I_1 = xe^x - e^x$.
4. Substitute back to find I_3.

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5. Summary of Common Reduction Formulae

1. **Powers of Sine:**

$$I_n = -\frac{\cos(x)\sin^{n-1}(x)}{n} + \frac{n-1}{n} I_{n-2}.$$

2. **Powers of Exponential:**

$$I_n = x^n e^x - nI_{n-1}.$$

3. **Powers of Tangent:**

$$I_n = \frac{\tan^{n-1}(x)}{n-1} - \frac{1}{n-1} I_{n-2}.$$

4. **Powers of Cosine:**

$$I_n = \frac{\sin(x)\cos^{n-1}(x)}{n} + \frac{n-1}{n} I_{n-2}.$$

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6. Advantages of Reduction Formulae

- Simplifies higher-order integrals.
- Recursively reduces complexity to basic integrals.
- Essential in solving complex problems in engineering, physics, and applied mathematics.