# Saitechinfo NEET-JEE Academy

## Improper Integrals: Lecture Notes

Improper integrals arise when we attempt to evaluate definite integrals where one or more of the following occurs:

1. The interval of integration is infinite.
2. The integrand becomes unbounded (infinite) within the interval.

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## 1. Definition

An integral is called **improper** if it involves one or both of the following:

- **Type 1: Infinite Limits of Integration**
  When the integration is over an infinite interval, such as $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^\infty f(x)\,dx$.
- **Type 2: Discontinuous Integrand**
  When the integrand $f(x)$ has a vertical asymptote at some point in the interval $[a,b]$, such as $\int_a^b f(x)\,dx$ where $f(x) \to \infty$ as $x \to c$ (for some $c \in [a,b]$).

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## 2. Evaluating Improper Integrals

### Type 1: Infinite Limits of Integration

To evaluate $\int_a^\infty f(x)\,dx$:

1. Replace $\infty$ with a variable $t$ and take the limit as $t \to \infty$:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

Example: Evaluate $\int_1^\infty \frac{1}{x^2}\,dx$.

1. Write the integral with a limit:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

2. Find the antiderivative of $\frac{1}{x^2}$:

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

3. Evaluate the definite integral:

$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = -\frac{1}{t} + \frac{1}{1}$$

4. Take the limit as $t \to \infty$:

$$\lim_{t\to\infty}\left(-\frac{1}{t} + 1\right) = 1$$

Result:

$$\int_1^\infty \frac{1}{x^2}\,dx = 1$$

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**Type 2: Discontinuous Integrand**

To evaluate $\int_a^b f(x)\,dx$, where $f(x)$ is undefined or infinite at a point $c \in [a, b]$:

1. Split the integral at the discontinuity:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

2. Replace the point of discontinuity $c$ with a limit and evaluate:

$$\int_a^c f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx, \quad \int_c^b f(x)\,dx = \lim_{t\to c^+}\int_t^b f(x)\,dx$$

Example: Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

1. The integrand $\frac{1}{\sqrt{x}}$ becomes infinite as $x \to 0^+$. Write the integral with a limit:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx$$

2. Find the antiderivative of $\frac{1}{\sqrt{x}}$:

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}$$

3. Evaluate the definite integral:

$$\int_t^1 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_t^1 = 2\sqrt{1} - 2\sqrt{t}$$

4. Take the limit as $t \to 0^+$:

$$\lim_{t\to 0^+}\left(2 - 2\sqrt{t}\right) = 2$$

Result:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$$

**Both Types: Infinite Limits and Discontinuity**

Improper integrals can involve both infinite limits and discontinuities.

Example: Evaluate $\int_1^\infty \frac{1}{x\ln(x)}\,dx$.

1. Note that the integrand $\frac{1}{x\ln(x)}$ is undefined at $x = 1$ and has an infinite limit at $\infty$.
2. Write the integral with limits:

$$\int_1^\infty \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x\ln(x)}\,dx$$

3. Use substitution. Let $u = \ln(x)$, $du = \frac{1}{x}dx$:

$$\int \frac{1}{x\ln(x)}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

4. Transform the limits for $u$:
   - When $x = 1$, $u = \ln(1) = 0$.
   - When $x = t$, $u = \ln(t)$.
5. Substitute back:

$$\int_1^t \frac{1}{x\ln(x)}\,dx = [\ln(\ln(x))]_1^t = \ln(\ln(t)) - \ln(\ln(1))$$

6. As $\ln(1) = 0$, this integral diverges.

Result:

$$\int_1^\infty \frac{1}{x\ln(x)}\,dx \text{ diverges.}$$

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## 3. Convergence and Divergence

- An improper integral **converges** if the corresponding limit exists and is finite.
- It **diverges** if the limit does not exist or is infinite.

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## 4. Key Examples

1. **Infinite Interval:** $\int_0^\infty e^{-x}\,dx$
   - Converges to $1$.
2. **Discontinuity:** $\int_1^2 \frac{1}{(x-1)^2}\,dx$
   - Diverges because $\frac{1}{(x-1)^2}$ becomes infinite at $x = 1$.
3. **Combined Case:** $\int_0^\infty \frac{1}{x^2+1}\,dx$
   - Converges to $\frac{\pi}{2}$.

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## 5. Applications

1. **Physics:** Calculating probabilities in quantum mechanics using normalized wavefunctions.
2. **Probability:** Evaluating cumulative distributions in statistics.
3. **Engineering:** Solving Laplace transforms in signal processing.