Saitechinfo NEET-JEE Academy

Gamma Integral: Lecture Notes

The **Gamma function**, denoted as $\Gamma(z)$, is a special function that generalizes the concept of factorials to real and complex numbers (excluding negative integers). It is widely used in mathematics, physics, and engineering.

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1. Definition of Gamma Function

The Gamma function is defined as an improper integral for $z > 0$:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

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2. Properties of the Gamma Function

1. **Recurrence Relation**:

$$\Gamma(z+1) = z\Gamma(z)$$

   - This property shows the connection between $\Gamma(z)$ and $\Gamma(z+1)$, similar to the factorial relationship.

2. **Gamma and Factorials**: For positive integers n,

$$\Gamma(n) = (n-1)!$$

   - Example: $\Gamma(4) = 3! = 6$.

3. **Gamma of Half-Integers**:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}, \ldots$$

4. **Multiplication Theorem**: For $z > 0$,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

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3. Derivation and Evaluation

Case 1: $\Gamma(1)$

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t} \, dt = \int_0^\infty e^{-t} \, dt$$

Evaluate:

$$\Gamma(1) = \left[-e^{-t}\right]_0^\infty = 1$$

Case 2: $\Gamma(2)$

Using the recurrence relation:

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1! = 1$$

Case 3: $\Gamma(3)$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2! = 2$$

Case 4: $\Gamma\left(\frac{1}{2}\right)$

Start with the integral definition:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$$

Use the substitution $t = u^2$, $dt = 2u\, du$:

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty (u^2)^{-\frac{1}{2}} e^{-u^2} (2u\, du)$$

Simplify:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du$$

Recognize this as the Gaussian integral:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

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4. Examples

Example 1: Evaluate $\Gamma(5)$

Using the recurrence relation:

$$\Gamma(5) = 4 \cdot \Gamma(4) = 4 \cdot 3 \cdot \Gamma(3) = 4 \cdot 3 \cdot 2 \cdot \Gamma(2) = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example 2: Evaluate $\Gamma\left(\frac{3}{2}\right)$

Using the recurrence relation:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

Substitute $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$:

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Example 3: Evaluate $\Gamma\left(\frac{5}{2}\right)$

Using the recurrence relation:

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right)$$

Substitute $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$:

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

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5. Applications of Gamma Function

1. **Probability and Statistics**:
   - Used in the definition of the gamma and beta distributions.
   - Appears in the normal distribution when handling moments or likelihood functions.
2. **Physics**:
   - Used in quantum mechanics, wave equations, and modeling decay processes.
   - Appears in computations involving factorial-like terms for non-integer values.
3. **Engineering**:
   - Applied in signal processing and solving differential equations.

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6. Exercises

1. Prove the recurrence relation $\Gamma(z+1) = z\Gamma(z)$.
2. Evaluate $\Gamma(6)$ using the recurrence relation.
3. Show that $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$ using integration.