Saitechinfo NEET-JEE Academy

Evaluation of Bounded Plane Area by Integration

Integration provides a powerful method to calculate the area of a region enclosed by curves in a plane. The process involves setting up definite integrals based on the boundaries of the region and integrating the difference of functions.

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1. General Method for Finding Bounded Areas

The area of a region bounded by curves can be evaluated using the following steps:

Case 1: Vertical Slices (Integrating with respect to x)

1. **Identify the boundaries:**
   - The region is bounded by curves $y = f(x)$ (upper curve) and $y = g(x)$ (lower curve) over the interval $[a, b]$.
2. **Set up the integral:**
   - The area is:

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx$$

3. **Integrate:**
   - Solve the definite integral to find the area.

Case 2: Horizontal Slices (Integrating with respect to y)

1. **Identify the boundaries:**
   - The region is bounded by $x = f(y)$ (right curve) and $x = g(y)$ (left curve) over the interval $[c, d]$.
2. **Set up the integral:**
   - The area is:

$$A = \int_c^d \big(f(y) - g(y)\big)\, dy$$

3. **Integrate:**
   - Solve the definite integral to find the area.

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2. Special Cases

Case 1: Area Between Two Curves

For the region enclosed by $y = f(x)$ and $y = g(x)$ over $[a, b]$:

$$A = \int_a^b \left| f(x) - g(x) \right| dx$$

Case 2: Symmetry

For symmetric regions about the x- or y-axis, calculate the area for one side and double it:

$$A = 2 \int_0^a f(x)\, dx \quad \text{(if symmetric about } y = 0\text{)}.$$

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3. Examples

Example 1: Area Between Two Curves

Find the area enclosed by $y = x^2$ and $y = x + 2$.

1. **Identify intersection points:**
   - Solve $x^2 = x + 2$:

     $$x^2 - x - 2 = 0 \quad \Rightarrow \quad (x - 2)(x + 1) = 0$$

     Intersection points are $x = -1$ and $x = 2$.
2. **Set up the integral:**
   - For $x \in [-1, 2]$, $y = x + 2$ is above $y = x^2$:

     $$A = \int_{-1}^{2} \left((x + 2) - x^2 \right) dx$$

3. **Integrate:**

   $$A = \int_{-1}^{2} (x + 2 - x^2)\, dx = \int_{-1}^{2} x\, dx + \int_{-1}^{2} 2\, dx - \int_{-1}^{2} x^2\, dx$$

   Compute each term:

   $$\int_{-1}^{2} x\, dx = \left[\frac{x^2}{2} \right]_{-1}^{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2},$$

   $$\int_{-1}^{2} 2\, dx = [2x]_{-1}^{2} = 4 - (-2) = 6,$$

   $$\int_{-1}^{2} x^2\, dx = \left[\frac{x^3}{3} \right]_{-1}^{2} = \frac{8}{3} - \left(-\frac{1}{3} \right) = \frac{9}{3} = 3.$$

   Combine results:

   $$A = \frac{3}{2} + 6 - 3 = \frac{9}{2}.$$

4. **Final result:**

$$A = \frac{9}{2}.$$

Example 2: Area Between a Curve and the x-Axis

Find the area enclosed by $y = x^3 - x^2$ and the x-axis.

1. **Identify intersection points:**
   - Solve $x^3 - x^2 = 0$:

   $$x^2(x-1) = 0 \quad \Rightarrow \quad x = 0,\ 1.$$

2. **Set up the integral:**
   - Split the region at $x = 0$ and $x = 1$, as y changes sign:

   $$A = \int_0^1 \left| x^3 - x^2 \right| dx$$

3. **Integrate:**
   - For $x \in [0, 1]$, $x^3 - x^2 \leq 0$:

   $$A = -\int_0^1 (x^3 - x^2)\, dx$$

   Compute:

   $$A = -\int_0^1 x^3\, dx + \int_0^1 x^2\, dx$$

   $$A = -\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1 = -\frac{1}{4} + \frac{1}{3}.$$

   Simplify:

   $$A = \frac{-3}{12} + \frac{4}{12} = \frac{1}{12}.$$

4. **Final result:**

$$A = \frac{1}{12}.$$

Example 3: Symmetric Region

Find the area enclosed by $y = \sqrt{1 - x^2}$ (a semicircle).

1. **Set up the integral:**
   - Since the region is symmetric about the y-axis, evaluate for $x \in [0, 1]$:

$$A = 2\int_0^1 \sqrt{1-x^2}\,dx$$

2. **Substitution:**
   - Let $x = \sin(\theta)$, $dx = \cos(\theta)\,d\theta$, $\sqrt{1-x^2} = \cos(\theta)$.
   - Bounds: $x = 0 \to \theta = 0$, $x = 1 \to \theta = \pi/2$.

   Substitute:

$$A = 2\int_0^{\pi/2} \cos^2(\theta)\,d\theta.$$

3. **Simplify:**
   - Use the identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$:

$$A = 2\int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\,d\theta = \int_0^{\pi/2} 1\,d\theta + \int_0^{\pi/2} \cos(2\theta)\,d\theta.$$

4. **Compute:**
   - First term:

$$\int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}.$$

   - Second term:

$$\int_0^{\pi/2} \cos(2\theta)\,d\theta = \frac{\sin(2\theta)}{2}\Big|_0^{\pi/2} = 0.$$

   Combine:

$$A = \frac{\pi}{2}.$$

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4. Applications

1. **Physics:**
   - Calculating areas under velocity-time graphs to determine displacement.
2. **Engineering:**
   - Determining cross-sectional areas for beams or structures.
3. **Economics:**
   - Evaluating consumer surplus and producer surplus.

Mastering these techniques equips you to calculate areas efficiently, a critical skill in many applied disciplines.