Saitechinfo NEET-JEE Academy

Definite Integral as the Limit of a Sum

The definite integral is a fundamental concept in calculus that connects the discrete summation of areas under a curve with continuous accumulation. Here's a structured lecture note on this concept:

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1. Introduction to Definite Integrals

The definite integral of a function $f(x)$ over an interval $[a, b]$ represents the net area under the curve $y = f(x)$ from $x = a$ to $x = b$.

$$\int_a^b f(x)\,dx$$

This integral can be interpreted as the limit of a sum of areas of rectangles as the number of rectangles approaches infinity.

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2. Approximating Area Using Riemann Sums

To approximate the area under the curve:

1. Divide the interval $[a, b]$ into n subintervals of equal width:

$$\Delta x = \frac{b-a}{n}$$

2. In each subinterval, choose a point x_i^* (e.g., left endpoint, right endpoint, or midpoint).
3. Compute the sum of the areas of rectangles formed by $f(x_i^*)$ (height) and Δx (width):

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

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3. Limit of Riemann Sum

As $n \to \infty$, the width of the rectangles (Δx) approaches zero, and the sum converges to the exact area under the curve:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

This is the definition of the definite integral as a limit of a Riemann sum.

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4. Examples

Example 1: Compute the Definite Integral

Find $\int_0^1 x^2\,dx$ using the definition of a definite integral.

1. Divide $[0,1]$ into n subintervals:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

2. Use right endpoints: $x_i^* = \frac{i}{n}$ for $i = 1, 2, \ldots, n$.
3. The Riemann sum becomes:

$$S_n = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\Delta x = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

4. Simplify:

$$S_n = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

5. Use the formula for the sum of squares:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

6. Substitute and simplify:

$$S_n = \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

7. Take the limit as $n \to \infty$:

$$\int_0^1 x^2\,dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}$$

Example 2: Verify $\int_1^2 3x^2\,dx$

1. Divide $[1,2]$ into n subintervals:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n}$$

2. Use left endpoints: $x_i^* = 1 + \frac{i-1}{n}$ for $i = 1, 2, \ldots, n$.
3. The Riemann sum becomes:

$$S_n = \sum_{i=1}^{n} f(x_i^*)\Delta x = \sum_{i=1}^{n} 3\left(1+\frac{i-1}{n}\right)^2 \cdot \frac{1}{n}$$

4. Expand $\left(1+\frac{i-1}{n}\right)^2$, simplify the sum, and take the limit as $n \to \infty$.

   Result:

$$\int_1^2 3x^2\,dx = \left[x^3\right]_1^2 = 2^3 - 1^3 = 8 - 1 = 7$$

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5. Key Points

1. The definite integral calculates the exact area under a curve by taking the limit of Riemann sums.
2. Riemann sums depend on the choice of x_i^*, but all approaches lead to the same integral value as $n \to \infty$.

These examples illustrate the concept and method of evaluating definite integrals using the limit of a sum.