Saitechinfo NEET-JEE Academy

Bernoulli’s Formula

Bernoulli’s Formula is a technique used to evaluate integrals involving products of a polynomial and another function that is easily integrable, such as e^x, $\sin(x)$, or $\cos(x)$. It leverages the method of integration by parts repeatedly.

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1. Statement of Bernoulli’s Formula

For a function $y(x) = P(x)Q(x)$, where $P(x)$ is a polynomial and $Q(x)$ is a function that is continuously differentiable, Bernoulli's Formula is derived from integration by parts:

$$\int P(x)Q(x)\,dx = P(x)\int Q(x)\,dx - \int P'(x)\left(\int Q(x)\,dx\right)\,dx$$

Expanded Form:

$$\int P(x)Q(x)\,dx = P(x)R(x) - P'(x)R(x) + P''(x)R(x) - \cdots + (-1)^n P^{(n)}(x)R(x) + (-1)^{n+1}\int$$

where:

- $R(x) = \int Q(x)\,dx$ (the antiderivative of $Q(x)$),
- $P^{(n)}(x)$ is the n-th derivative of $P(x)$.

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2. Simplification Using Polynomials

Since $P(x)$ is a polynomial, its derivatives will eventually become zero after a finite number of steps. This makes Bernoulli’s formula particularly useful when $P(x)$ is a polynomial.

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3. Applications of Bernoulli’s Formula

Case 1: Polynomial $\times$ Exponential

For $\int P(x)e^x\,dx$, the exponential function e^x remains the same upon integration.

Case 2: Polynomial $\times$ Trigonometric

For $\int P(x)\sin(x)\,dx$ or $\int P(x)\cos(x)\,dx$, integration by parts will alternate between $\sin(x)$ and $\cos(x)$.

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4. Examples

Example 1: Evaluate $\int xe^x\,dx$

1. Let $P(x) = x$ and $Q(x) = e^x$.
2. Using integration by parts:

$$\int xe^x \, dx = x \int e^x \, dx - \int \frac{d}{dx}(x) \cdot \int e^x \, dx$$

3. Simplify:

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

4. Compute:

$$\int xe^x \, dx = xe^x - e^x + C$$

5. Final result:

$$\int xe^x \, dx = e^x(x-1) + C$$

Example 2: Evaluate $\int x^2 e^x \, dx$

1. Let $P(x) = x^2$ and $Q(x) = e^x$.
2. First integration by parts:

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx$$

3. For $\int 2xe^x \, dx$, apply integration by parts again:

$$\int 2xe^x \, dx = 2\left(xe^x - \int e^x \, dx\right) = 2(xe^x - e^x)$$

4. Substitute back:

$$\int x^2 e^x \, dx = x^2 e^x - 2(xe^x - e^x)$$

5. Simplify:

$$\int x^2 e^x \, dx = e^x(x^2 - 2x + 2) + C$$

Example 3: Evaluate $\int x \sin(x) \, dx$

1. Let $P(x) = x$ and $Q(x) = \sin(x)$.
2. Using integration by parts:

$$\int x \sin(x) \, dx = -x \cos(x) + \int \cos(x) \, dx$$

3. Compute:

$$\int x\sin(x)\,dx = -x\cos(x) + \sin(x) + C$$

Example 4: Evaluate $\int x^2\cos(x)\,dx$

1. Let $P(x) = x^2$ and $Q(x) = \cos(x)$.
2. First integration by parts:

$$\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx$$

3. For $\int 2x\sin(x)\,dx$, apply integration by parts:

$$\int 2x\sin(x)\,dx = -2x\cos(x) + \int 2\cos(x)\,dx = -2x\cos(x) + 2\sin(x)$$

4. Substitute back:

$$\int x^2\cos(x)\,dx = x^2\sin(x) - (-2x\cos(x) + 2\sin(x))$$

5. Simplify:

$$\int x^2\cos(x)\,dx = x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C$$

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5. Key Steps in Applying Bernoulli's Formula

1. Identify $P(x)$ (polynomial) and $Q(x)$ (exponential or trigonometric function).
2. Apply integration by parts repeatedly until the polynomial $P(x)$ reduces to zero.
3. Simplify and combine terms at each step.
4. Add the constant of integration at the end.

Advantages of Bernoulli's Formula

- Streamlines the process of evaluating integrals with a polynomial and a non-polynomial factor.
- Efficient for repetitive integration by parts.

This technique is especially valuable in mathematical physics, engineering, and higher mathematics where such integrals frequently appear.