Solution

DIFFERENTIAL EQUATIONS

Class 12 - Mathematics

1. y = cos x + c

   y^1 = -sin x

   y^1 + sin x = 0
2. It is given that equation is y’’ + $(y')^2$ + 2y = 0

   We can see that the highest order derivative present in the differential is y”.

   Thus, its order is two.

   Highest power raised to y” is 1

   Therefore, its degree is one.
3. The order of this differential equation is 2, because the highest order derivative in is second order.

   The degree is the power of this highest order derivative. In this case degree is 2.

   So, the answer is 2 × 2 = 4.
4. $\frac{dy}{dx}$ = (e^x + 1) y

   $\Rightarrow \frac{1}{y}$ dy = (e^x + 1) dx

   Integrating both sides, we obtain

   $\int \frac{1}{y} dy = \int (e^x + 1) dx$

   $\Rightarrow$ log |y| = e^x + x + C

   Hence, log |y| = e^x + x + C is the required solution.
5. Since the equation has the power of highest derivative 2 .Therefore its degree is 2.
6. Given differential equation can be written as

   $\frac{dy}{dx} + \left(1 - \frac{1}{x}\right) y = \frac{1}{x}$

   Therefore integrating factor = $e^{x - \log x}$ or $\frac{e^x}{x}$
7. Given $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

   $\Rightarrow \log y = \log x + \log c \Rightarrow log\ y = \log\ cx$

   $\Rightarrow y = cx$
8. Consider the given differential equation,

   $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0$

   Then the degree of a differential equation is the degree of the highest order derivative.

   In this given differential equation, the power of the highest order differential coefficient is 3 .

   Therefore the degree of this diff equation is 3.
9. Given differential equation is

   $\frac{dy}{dx} = x^3 e^{-2y}$

   On separating the variables, we get

   e^{2y}dy = x^3dx

   On integrating both sides, we get

   $\int e^{2y} dy = \int x^3 dx$

   $\Rightarrow \quad \frac{e^{2y}}{2} = \frac{x^4}{4} + C_1$

   $\Rightarrow 2e^{2y} = x^4 + 4C_1$

   $\therefore \quad 2e^{2y} = x^4 + C$, where C = 4 C_1
10. The given diff. equation is not free from radical sign. Therefore Degree is not defined.
11. The given differential equation is,

    $\frac{dy}{dx}$ + 2y = sin x

    comparing with $\frac{dy}{dx}$ + Py = Q,

    P = 2, and Q = sin x

Now, I.F. = $e^{\int 2dx}$ = e^{2x}

$\therefore$ y.I.F. = $Q.I.Fdx + c$

$\Rightarrow$ y.2e^{x} = $\int sinxe^{2x}dx + c$...(i)

Let I = $\int sinxe^{2x}$

$= sinx \int e^{2x}dx$ - $\int cos\frac{e^{2x}}{2}dx$

$= \frac{1}{2}sinx\frac{e^{2x}}{2}$ - $\int cos\frac{e^{2x}}{2}dx$

$= \frac{1}{2}sinxe^{2x}$ - $\frac{1}{2}[cos\int e^{2x} + \int sinxdx[\frac{e^{2x}}{2}dx]$

$= \frac{1}{2}sinxe^{2x}$ - $\frac{1}{2}[cosx\frac{\int e^{2x}}{2} + \frac{1}{2}\int sinxdx[e^{2x}dx]$

$= \frac{1}{2}sinxe^{2x}$ - $\frac{1}{2}[cosxe^{2x} + \frac{1}{2}I] + c$

$\Rightarrow$ I = $\frac{1}{2}sinxe^{2x}$ - $\frac{1}{4}cosxe^{2x}$ - $\frac{1}{4}I$ + c

$\Rightarrow$ $\frac{5}{4}I$ = $\frac{e^{2x}}{4}$ [2 sinx - cos x] + c

$\Rightarrow$ I = $\frac{e^{2x}}{5}$ [2 sinx - cosx] + c

Using this value in equation (i),we get,

y.e^{2x} = $\frac{e^{2x}}{5}$[2 sin x - cos x] + c

12. According to the question, $\frac{dy}{dx}$+ 2ytanx = sinx

Given equation is a linear differential equation in the form $\frac{dy}{dx}$+ Py = Q

Here, P = 2tanx and Q = sinx

Now, Integration Factor(IF) = $e^{\int Pdx} = e^{\int 2\tan xdx}$

$= e^{2\int \tan xdx}$

$= e^{2\log|\sec x|}$

$= e^{\log \sec^2 x}$

$= \sec^2 x \left[\because e^{\log f(x)} = f(x)\right]$

The solution of differential equation is given by

$y \cdot (IF) = \int (\text{IF}) Qdx + C$

$\Rightarrow y \cdot \sec^2 x = \int \sec^2 x \cdot \sin xdx + C$

$\Rightarrow y \cdot \sec^2 x = \int \frac{\sin x}{\cos^2 x}dx + C$

$\Rightarrow y \cdot \sec^2 x = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}dx + C$

$\Rightarrow y \cdot \sec^2 x = \int \tan x \cdot \sec xdx + C$

$\Rightarrow y \cdot \sec^2 x = \sec x + C$

Dividing with sec^{2}x on both sides, we get

$\Rightarrow y = \frac{1}{\sec x} + \frac{C}{\sec^2 x}$

$\Rightarrow$y = cosx + C.cos^{2}x

According to the question, y = 0 when x = $\frac{\pi}{3}$

$0 = \cos\frac{\pi}{3} + C \cdot \cos^2\frac{\pi}{3}$

$\Rightarrow 0 = \frac{1}{2} + C \cdot \frac{1}{4}$

$\Rightarrow \frac{-1}{2} = \frac{C}{4}$

$\Rightarrow C = -2$

Now, y = cosx + C.cos^{2}x $\Rightarrow$ y = cos x - 2 cos^{2}x

Therefore, the required particular solution is y = cos x - 2 cos^{2}x

13. We have,

x$\frac{dy}{dx}$ + 2y = x^{2}log x

$\Rightarrow$ $\frac{dy}{dx}$ + $\frac{2}{x}$ y = x log x

This equation is in the form of a linear differential equation as:

$\frac{dy}{dx}$ + py = Q (where P = $\frac{2}{x}$ and Q = x log x)

Now,we have,

I.F = $e^{\int pdx} = e^{\int \frac{2}{x}dx}$ = e$^{2\log x}$ $= e^{\log x^2}$ =x^{2}.

The general solution of the given differential equation is given by the relation,

y(I.F.) = $\int$(Q $\times$ I.F) dx + C

$\Rightarrow$ y x^2 = $\int$(x log x . x^2) dx + C

$\Rightarrow$ x^2y = $\int$(x^3 log x) dx + C

$\Rightarrow$ x^2y = log x $\int x^3 dx - \int \left[\frac{d}{dx}(\log x) \cdot \int x^3 dx\right]$ dx + C

$\Rightarrow$ x^2y = log x .$\frac{x^4}{4} - \int \left(\frac{1}{x} \cdot \frac{x^4}{4}\right)$dx + C

$\Rightarrow$ x^2y = $\frac{x^4 \log x}{4} - \frac{1}{4} \int x^3 dx + \mathrm{C}$

$\Rightarrow$ x^2y = $\frac{x^4 \log x}{4} - \frac{1}{4} \cdot \frac{x^4}{4} + \mathrm{C}$

$\Rightarrow$ x^2y = $\frac{1}{16} x^4$ (4log x - 1) + C

$\Rightarrow$ y = $\frac{1}{16} x^2$ (4log x - 1) + Cx^{-2}

14. Given that, interest is compounded 6% per annum. Let p be principal

$\frac{dP}{dt} = \frac{\mathrm{Pr}}{100}$

$\frac{dP}{dt} = \frac{r}{100} dt$

$\int \frac{dP}{P} = \int \frac{r}{100} dt$

$\log P = \frac{rt}{100} + c$ ....(1)

Let P_0 be the initial principal at t = 0

$\log(P_0) = 0 + c$

$c = \log(P_0)$

Put value of C is equation (1),then,we have,

$\log(p) = \frac{rt}{100} + \log(p_0)$

$\log\left(\frac{p}{p_0}\right) = \frac{rt}{100}$

Case 1:

Here, P_0 = 1000, t = 10 years and r = 6

$\log\left(\frac{p}{1000}\right) = \frac{6 \times 10}{100}$

$\log p - \log 1000 = 0.6$

$\log p = \log e^{0.6} + \log 1000$

$= \log\left(e^{0.6} + 1000\right)$

$= \log(1.822 + 1000)$

log P = log 1822

Therefore,

P = ₹1822

Rs 1000 will be R s 1822 after 10 years

Case 2: let t_1 be the time to double ₹ 1000, therefore,we have,

P = 2000, P_0 = 1000, r = 696

$\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$

$\log\left(\frac{2000}{1000}\right) = \frac{6t_1}{100}$

$\frac{100 \log 2}{6} = t_1$

$\frac{100 \times 0.6931}{6} = t_1$

11.55 years $= t_1$

It will take approximately 12 years to double

15. The given differential equation is,

$\sin^{-1}\left(\frac{dy}{dx}\right)$= x + y

$\Rightarrow \frac{dy}{dx}$ = sin (x + y)

Let x + y = v. Then,

$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Put, x + y = v and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$\therefore \frac{dv}{dx}$ - 1 = sin v

$\Rightarrow \frac{dv}{dx}$ = 1 + sin v

$\Rightarrow \frac{1}{1+\sin v}$ dv = dx

$\Rightarrow \int \frac{1}{1+\sin v} dv = \int dx$ [Integrating both sides]

$\Rightarrow \int dx = \int \frac{1-\sin v}{1-\sin^2 v} dv$

$\Rightarrow \int dx = \int \frac{1-\sin v}{\cos^2 v} dv$

$\Rightarrow \int$ dx = $\int$ ($\sec^2$ v - tan v sec v) dv

$\Rightarrow$ x = tan v - sec v + C

$\Rightarrow$ x = tan (x + y) - sec (x + y) + C, which is the required solution.

16. (i) Let the population at any instant (t) be y.

Now it is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$\therefore \frac{dy}{dt} \alpha y$

$\Rightarrow \frac{dy}{dt} = ky$ (k is constant)

$\Rightarrow \frac{dy}{y} = kdt$

Now, integrating both sides, we get,

log y = kt + C ....(i)

According to given conditions,

In the year 1999, t = 0 and y = 20000

$\Rightarrow$ log20000 = C .....(ii)

Also, in the year 2004, t = 5 and y = 25000

$\Rightarrow$ log 25000 = k.5 + C

$\Rightarrow$ log 25000 = 5k + log 20000

$\Rightarrow 5\mathrm{k} = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow \mathrm{k} = \frac{1}{5}\log\left(\frac{5}{4}\right)$ ......(iii)

Also, in the year 2009, t = 10

Now, substituting the values of t, k and c in equation (i), we get

$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$

$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$

$\Rightarrow \mathrm{y} = 20000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow$ y = 31250

Therefore, the population of the village in 2009 will be 31250.

(ii) Let the population at any instant (t) be y.

By the given condition,we have,

$\frac{dy}{dt} \alpha y$

$\Rightarrow \frac{dy}{dt} = ky$ (k is constant)

$\Rightarrow \frac{dy}{y} = kdt$

Now, integrating both sides, we get,

log y = kt + C ...(i)

According to given conditions,

In the year 1990, t = 0 and y = 200000

$\Rightarrow$ log200000 = C ...(ii)

Also, in the year 2000, t = 5 and y = 250000

$\Rightarrow$ log 250000 = k.5 + C

$\Rightarrow$ log 250000 = 5k + log 200000

$\Rightarrow 5\mathrm{k} = \log\left(\frac{250000}{200000}\right) = \log\left(\frac{5}{4}\right)$

$\Rightarrow \mathrm{k} = \frac{1}{5}\log\left(\frac{5}{4}\right)$ ......(iii)

Also, in the year 2010, t = 10

Now, substituting the values of t, k and c in equation (i), we get

$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(200000)$

$\Rightarrow \log y = \log\left[200000 \times \left(\frac{5}{4}\right)^2\right]$

$\Rightarrow$ y $= 200000 \times \frac{5}{4} \times \frac{5}{4}$

$\Rightarrow$ y = 312500

Therefore, the population of the village in 2010 will be 312500.

17. i. We have, $(1 - x^2)\frac{dy}{dx}$ - xy = 1

$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2} \cdot y = \frac{1}{1-x^2}$

$\therefore$ I.F. $= e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx}$

$= e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{\frac{1}{2}}} = \sqrt{1-x^2}$

ii. We have, $\frac{dy}{dx}$ + y = e^{-x}

It is a linear differential equation with I.F. $= e^{\int dx} = e^x$

Now, solution is $y \cdot e^x = \int e^x \cdot e^{-x}$ dx + c

$\Rightarrow ye^x = \int dx + c \Rightarrow ye^x = x + c \Rightarrow y = xe^{-x} + ce^{-x}$

$\because y(0) = 0 \Rightarrow c = 0 \therefore y = xe^{-x}$

iii. We have, $\frac{dy}{dx}$ + y tan x = sec x

It is a linear differential equation with

I.F. $= e^{\int \tan x dx} = e^{\log|\sec x|}$ = sec x

Now, solution is y secx = $\int \sec^2 x$ dx + c

$\Rightarrow$ y secx = tanx + c

OR

We have, $\frac{dy}{dx}$ - 3y = sin 2x

It is a linear differential equation with

I.F. $= e^{\int -3dx} = e^{-3x}$

18. (i) Let A be the quantity of radium present at time t and A_0 be the initial quantity of radium.

Therefore,we have,

$\frac{dA}{dt} \propto A$

$\frac{dA}{dt}$ = - 2A

$\frac{dA}{A}$ = - 2dt

$\int \frac{dA}{A} = -\lambda t dt$

$\log A = -\lambda t$ + c.....(1)

Now, A = A_0 when t = 0

log A_0 = 0 + c

c = log A_0

Put value of c in equation

log A = $-\lambda t + \log A_0$

$\log\left(\frac{A}{A_0}\right) = -\lambda t$ .....(2)

Given that, In 25 years, bacteria decomposes 1.1 %, so

A = (100 - 1.1)% = 98.996 %= 0.989 A_0, t = 25

Therefore,(2) gives,

$\log\left(\frac{0.989A_0}{A_0}\right) = -25\lambda$

$\log(0.989) = -25\lambda$

$\lambda = -\frac{1}{25}\log(0.989)$

Now, equation (2) becomes,

$\log\left(\frac{A}{A_0}\right) = \left\{\frac{1}{25}\log(0.989)\right\}t$

Now A = $\frac{1}{2}A_0$

$\log\left(\frac{A}{2A}\right) = \frac{1}{25}\log(0.989)t$

$\frac{-\log 2\times 25}{\log(0.989)}$ = t

$-\frac{0.6931\times 25}{0.01106}$ = t

t = 1567 years.

Required time = 1567 years

(ii) Let A be the quantity of radius at any time t, therefore,we have,

$\frac{dA}{dt}\propto A$

$\frac{dA}{dt}=-\lambda A$

$\frac{dA}{A}=-\lambda t$

$\int\frac{dA}{A}=-\lambda\int dt$

log A = $-\lambda$t + c ....(1)

Let A_0 be the initial amount of radium percentage,then,we have,

$\log A_0=-\lambda(0)$ + c

c = $\log(A_0)$

Using, equation (1),

log A = $-\lambda t$ + $\log A_0$

$\log\left(\frac{A}{A_0}\right)=-\lambda t$ ...(2)

Given, its half-life is 1590 years, therefore,we have,

$\log\left(\frac{\frac{1}{2}A_0}{A_0}\right)=-\lambda(1590)$

$\log\left(\frac{1}{2}\right)=-\lambda(1590)$

-log 2 = $-\lambda(1590)$

log 2 = $\lambda(1590)$

$\frac{\log 2}{1590}=\lambda$

Therefore, equation (1) becomes,

$\log\left(\frac{A}{A_0}\right)=-\frac{\log 2}{1590}t$

Now, put t=1,we have,

$\log\left(\frac{A}{A_0}\right)=-\frac{\log 2}{1590}$

$\frac{A}{A_0}=e^{-\frac{\log 2}{1590}}$

$\frac{A}{A_0}$ = 0.9996

$1-\frac{A}{A_0}$ = 1 - 0.9996

$\frac{A_0-A}{A_0}$ = 0.0004

percentage to be disappear is one year

$=\frac{A_0-A}{A_0}\times 100$

= 0.0004 $\times$ 100

=0.04%

19. i. Here, P denotes the principal at any time t and the rate of interest be r% per annum compounded continuously, then according to the law given in the problem, we get

    $\frac{dP}{dt}=\frac{Pr}{100}$

    ii. We have,

    $\frac{dP}{dt}=\frac{Pr}{100}$

    $\Rightarrow\frac{dP}{P}=\frac{r}{100}$ dt

    $\Rightarrow\int\frac{1}{P}dP=\frac{r}{100}\int$ dt

    $\Rightarrow$ log P = $\frac{rt}{100}$ + C

    At t = 0, P = P_0

    C = log p_0

    So, log P $\frac{rt}{100}$ + log p_0

    $\log\left(\frac{P}{P_0}\right)=\frac{rt}{100}$

iii. We have, P_0 = ₹ 100, P = ₹ 200 = $2P_0$ and t = 10 years

Substituting these values in log $\left(\frac{P}{P_0}\right) = \frac{rt}{100}$,

we get

$\log\left(\frac{2P_0}{P_0}\right) = \frac{10r}{100}$

$\Rightarrow$ log 2 = $\frac{10r}{100}$

$\Rightarrow$ r = 10 log 2

= 10 $\times$ 0.6931

= 6.931

OR

We have, P_0 = ₹ 1000, r = 5 and t = 10

Substituting these values in $\log\left(\frac{P}{P_0}\right) = \frac{rt}{100}$, we get

$\log\left(\frac{P}{1000}\right) = \frac{50}{100}$

$\Rightarrow \log\left(\frac{P}{1000}\right) = \frac{1}{2}$ = 0.5

$\Rightarrow \frac{P}{1000} = e^{0.5}$

$\Rightarrow$ P = 1000 $\times$ $e^{0.5}$

= 1000 $\times$ 1.648

= ₹ 1648

20. Let A be the amount of bacteria present at time t and A_0 be the initial amount of bacteria. Therefore,we have,

$\frac{dA}{dt} \propto A$

$\frac{dA}{dt} = \lambda A$

$\frac{dA}{A} = \lambda dt$

Integrating both sides,we get,

$\log A = \lambda t + c$ ...(1)

when t = 0,$A = A_0$

$log\,(A_0)$ = 0 + c

c = log A_0

Using equation (1),

log A = λt + log A_0

$\log\left(\frac{A}{A_0}\right) = \lambda t$ ..(2)

Given, bacteria triples is 5 hours, so A = 3 A_0, when t = 5,

therefore from (2),we have,

$\log\left(\frac{3A_0}{A_0}\right) = 5\lambda$

log3 = 5λ

$\lambda = \frac{\log 3}{5}$

Put the value of λ in equation (2),we have,

$\log\left(\frac{A}{A_0}\right) = \frac{\log 3}{5}t$

Case I: let A_1 be the number of bacteria present in 10 hours, then,we have,

$\log\left(\frac{A_1}{A_0}\right) = \frac{\log 3}{5} \times 10$

$\log\left(\frac{A_1}{A_0}\right) = 2\log 3$

$\log\left(\frac{A_1}{A_0}\right) = 2(1.0986)$

$\log\left(\frac{A_1}{A_0}\right)$ = 2.1972

$A_1 = A_0 e^{2.1972}$

$A_1 = A_0 9$

Hence,there will be 9 times the bateria present is 10 hours.

Case II: Let t_1 be the time necessary for the bacteria to be 10 times, then,we have,

$\log\left(\frac{A}{A_0}\right) = \frac{\log 3}{5} \times t$

$\log\left(\frac{10A_0}{A_0}\right) = \frac{\log 3}{5} \times t_1$

5 log 10 = log 3 t_1

$5\frac{\log 10}{\log 3} = t_1$

Required time is $\frac{5 \log 10}{\log 3}$ hours