

## 10.4 COHERENT AND INCOHERENT ADDITION OF WAVES

In this section we will discuss the interference pattern produced by the superposition of two waves. You may recall that we had discussed the superposition principle in Chapter 15 of your Class XI textbook. Indeed the entire field of interference is based on the *superposition principle* according to which *at a particular point in the medium, the resultant displacement produced by a number of waves is the vector sum of the displacements produced by each of the waves.*

Consider two needles  $S_1$  and  $S_2$  moving periodically up and down in an identical fashion in a trough of water [Fig. 10.8(a)]. They produce two water waves, and at a particular point, the phase difference between the displacements produced by each of the waves does not change with time; when this happens the two sources are said to be *coherent*. Figure 10.8(b) shows the position of crests (solid circles) and troughs (dashed circles) at a given instant of time. Consider a point P for which

$$S_1 P = S_2 P$$

Since the distances  $S_1 P$  and  $S_2 P$  are equal, waves from  $S_1$  and  $S_2$  will take the same time to travel to the point P and waves that emanate from  $S_1$  and  $S_2$  in phase will also arrive, at the point P, in phase.

Thus, if the displacement produced by the source  $S_1$  at the point P is given by

$$y_1 = a \cos \omega t$$

then, the displacement produced by the source  $S_2$  (at the point P) will also be given by

$$y_2 = a \cos \omega t$$

Thus, the resultant of displacement at P would be given by

$$y = y_1 + y_2 = 2 a \cos \omega t$$

Since the intensity is proportional to the square of the amplitude, the resultant intensity will be given by

$$I = 4 I_0$$

where  $I_0$  represents the intensity produced by each one of the individual sources;  $I_0$  is proportional to  $a^2$ . In fact at any point on the perpendicular bisector of  $S_1 S_2$ , the intensity will be  $4I_0$ . The two sources are said to

interfere constructively and we have what is referred to as *constructive interference*. We next consider a point Q [Fig. 10.9(a)] for which

$$S_2 Q - S_1 Q = 2\lambda$$

The waves emanating from  $S_1$  will arrive exactly two cycles earlier than the waves from  $S_2$  and will again be in phase [Fig. 10.9(a)]. Thus, if the displacement produced by  $S_1$  is given by

$$y_1 = a \cos \omega t$$

then the displacement produced by  $S_2$  will be given by

$$y_2 = a \cos (\omega t - 4\pi) = a \cos \omega t$$

where we have used the fact that a path difference of  $2\lambda$  corresponds to a phase difference of  $4\pi$ . The two displacements are once again in phase and the intensity will again be  $4 I_0$  giving rise to constructive interference. In the above analysis we have assumed that the distances  $S_1 Q$  and  $S_2 Q$  are much greater than  $d$  (which represents the distance between  $S_1$  and  $S_2$ ) so that although  $S_1 Q$  and  $S_2 Q$  are not equal, the amplitudes of the displacement produced by each wave are very nearly the same.

We next consider a point R [Fig. 10.9(b)] for which

$$S_2 R - S_1 R = -2.5\lambda$$

The waves emanating from  $S_1$  will arrive exactly two and a half cycles later than the waves from  $S_2$  [Fig. 10.10(b)]. Thus if the displacement produced by  $S_1$  is given by

$$y_1 = a \cos \omega t$$

then the displacement produced by  $S_2$  will be given by

$$y_2 = a \cos (\omega t + 5\pi) = -a \cos \omega t$$

where we have used the fact that a path difference of  $2.5\lambda$  corresponds to a phase difference of  $5\pi$ . The two displacements are now out of phase

and the two displacements will cancel out to give zero intensity. This is referred to as *destructive interference*.

To summarise: If we have two coherent sources  $S_1$  and  $S_2$  vibrating in phase, then for an arbitrary point P whenever the path difference,

$$S_1P - S_2P = n\lambda \quad (n = 0, 1, 2, 3, \dots) \quad (10.10)$$

we will have constructive interference and the resultant intensity will be  $4I_0$ ; the sign  $\sim$  between  $S_1P$  and  $S_2P$  represents the difference between  $S_1P$  and  $S_2P$ . On the other hand, if the point P is such that the path difference,

$$S_1P - S_2P = (n + \frac{1}{2})\lambda \quad (n = 0, 1, 2, 3, \dots) \quad (10.11)$$

we will have *destructive interference* and the resultant intensity will be zero. Now, for any other arbitrary point G (Fig. 10.10) let the phase difference between the two displacements be  $\phi$ . Thus, if the displacement produced by  $S_1$  is given by

$$y_1 = a \cos \omega t$$

then, the displacement produced by  $S_2$  would be

$$y_2 = a \cos (\omega t + \phi)$$

and the resultant displacement will be given by

$$\begin{aligned} y &= y_1 + y_2 \\ &= a [\cos \omega t + \cos (\omega t + \phi)] \\ &= 2 a \cos (\phi/2) \cos (\omega t + \phi/2) \\ &\left[ \because \cos A + \cos B = 2 \cos \left( \frac{A+B}{2} \right) \cos \left( \frac{A-B}{2} \right) \right] \end{aligned}$$

The amplitude of the resultant displacement is  $2a \cos (\phi/2)$  and therefore the intensity at that point will be

$$I = 4 I_0 \cos^2 (\phi/2) \quad (10.12)$$

If  $\phi = 0, \pm 2\pi, \pm 4\pi, \dots$  which corresponds to the condition given by Eq. (10.10) we will have constructive interference leading to maximum intensity. On the other hand, if  $\phi = \pm\pi, \pm 3\pi, \pm 5\pi \dots$  [which corresponds to the condition given by Eq. (10.11)] we will have destructive interference leading to zero intensity.

Now if the two sources are coherent (i.e., if the two needles are going up and down regularly) then the phase difference  $\phi$  at any point will not change with time and we will have a stable interference pattern; i.e., the positions of maxima and minima will not change with time. However, if the two needles do not maintain a constant phase difference, then the interference pattern will also change with time and, if the phase difference changes very rapidly with time, the positions of maxima and minima will also vary rapidly with time and we will see a "time-averaged" intensity distribution. When this happens, we will observe an average intensity that will be given by

$$\langle I \rangle = 4 I_0 \langle \cos^2 (\phi/2) \rangle \quad (10.13)$$

where angular brackets represent time averaging. Indeed it is shown in Section 7.2 that if  $\phi(t)$  varies randomly with time, the time-averaged quantity  $\langle \cos^2 (\phi/2) \rangle$  will be  $1/2$ . This is also intuitively obvious because the function  $\cos^2 (\phi/2)$  will randomly vary between 0 and 1 and the average value will be  $1/2$ . The resultant intensity will be given by

$$I = 2 I_0 \quad (10.14)$$

at all points.

When the phase difference between the two vibrating sources changes rapidly with time, we say that the two sources are incoherent and when this happens the intensities just add up. This is indeed what happens when two separate light sources illuminate a wall.