

## Solution

### RELATIONS AND FUNCTIONS

#### Class 12 - Mathematics

1. (a) An equivalence relation

**Explanation:**

$R = \{(P, Q): \text{distance of point P from the origin is the same as the distance of point Q from the origin}\}$

Let origin be O

Hence  $OP = OQ$

So,

$R = \{(P, Q): OP = OQ\}$

**Check reflexive**

since, point P and point P are same

Distance of Point P from origin = Distance of point P from the origin

So,  $(P, P) \in R$ ,

$\therefore R$  is reflexive.

**Check Symmetric**

If Distance of Point P from origin = Distance of point Q from the origin then,

Distance of Point Q from origin = Distance of point P from the origin

So, if  $(P, Q) \in R$ , then  $(Q, P) \in R$

$\therefore R$  is symmetric.

**Check transitive**

If Distance of Point P from origin = Distance of point Q from the origin & Distance of Point Q from origin = Distance of point S from the origin then

Distance of Point P from origin = Distance of point S from the origin

So, if  $(P, Q) \in R$  & if  $(Q, S) \in R$ , then  $(P, S) \in R$

$\therefore R$  is transitive.

Since R is reflexive, symmetric & transitive.

$\therefore R$  is an equivalence relation.

2. (a) Reflexive and symmetric but not transitive

**Explanation:**

Reflexivity: clearly for every  $a \in S$  (real number)

$$|a - a| = 0 < 1$$

$\Rightarrow R$  is Reflexive

Symmetric:  $(a, b) \in R$

$$\Rightarrow |a - b| < 1$$

$$\Rightarrow |b - a| < 1$$

$$\Rightarrow (b, a) \in R$$

$\therefore R$  is Symmetric

Transitive:  $(a, b), (b, c) \in R$

$$\Rightarrow |a - b| < 1 \text{ and } |b - c| < 1$$

$$\text{Now, } |a - c| = |a - b + b - c| \leq |a - b| + |b - c|$$

Which need not be less than 1

So, R is not Transitive

$\therefore R$  is Reflexive and symmetric but not transitive.

3. (a) one-one onto

**Explanation:**

$$f(x) = 2x^3 + 2x^2 + 300x + 5 \sin x$$

$$f'(x) = 6x^2 + 4x + 300 + 5 \cos x$$

$$= 2(3x^2 + 2x + 147) + (6 + 5\cos x)$$

Since  $-1 \leq \cos x \leq 1 \forall x \in \mathbf{R}$

$$\therefore 6 + 5 \cos x > 0$$

$$\text{let } g(x) = 3x^2 + 2x + 147$$

Since  $a = 3 > 0$  and  $D = (2)^2 - 4 \times 3 \times 147 < 0$

$$\therefore g(x) > 0$$

$$f'(x) > 0$$

$\therefore f(x)$  is an increasing function therefore said to be 1-1 function.

Now,  $f(-\infty) = -\infty$  and  $f(\infty) = \infty$

$\therefore f(x)$  is continuous also.

$\therefore$  Range of  $f =$  codomain of  $f = \mathbf{R}$

$\therefore f$  is onto.

4.

$$(d) g(x) = \sin\left(\frac{\pi x}{2}\right)$$

**Explanation:**

Given that  $A = \{x : -1 \leq x \leq 1\}$

$$f(x) = \frac{x}{2}$$

It is one-one but not onto.

$$g(x) = \sin\left(\frac{\pi x}{2}\right)$$

It is bijective as it is one-one and onto with range  $[-1, 1]$ .

$$h(x) = |x|$$

It is not one-one because  $h(-1) = 1$  and  $h(1) = 1$ .

$$k(x) = x^2$$

It is not one-one because  $k(-1) = 1$  and  $k(1) = 1$ .

5.

$$(d) \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

**Explanation:**

$$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2 \tan^{-1}x$$

$f(x)$  is one-one and onto

i.e.,  $f'(x) > 0$  or  $f'(x) < 0$  and co-domain = range of  $f(x)$

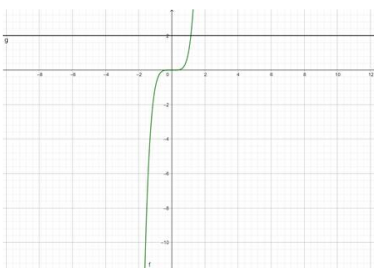
$$B = f(-1, 1) = (2 \tan^{-1}(-1), 2 \tan^{-1}(1))$$

$$= \left(2 \times \left(-\frac{\pi}{4}\right), 2 \times \frac{\pi}{4}\right) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

6. one - one and onto

$$f(x) = x^5$$

$$\Rightarrow y = x^5$$



Since the lines do not cut the curve in 2 equal valued points of  $y$ , therefore, the function  $f(x)$  is one - one.

The range of  $f(x) = (-\infty, \infty) = \mathbf{R}$  (Codomain)

$\therefore f(x)$  is onto

$\therefore f(x)$  is one - one and onto.

7. We have

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^3 = x_2^3 \\ &\Rightarrow (x_1^3 - x_2^3) = 0 \\ &\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \\ &\Rightarrow (x_1 - x_2) \left[ \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 \right] = 0 \\ &\Rightarrow (x_1 - x_2) = 0 \quad \left[ \cdot \cdot \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 \neq 0 \right] \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

f is one one.

Let  $y \in R$  and let  $y = x^3$ . Then,  $x = y^{1/3} \in R$

Thus, for each y in the codomain R there exists  $y^{1/3}$

Thus, for each y in the codomain R there exists y in R such that

$$f(y^{1/3}) = (y^{1/3})^3 = y$$

Hence, y is one-one onto.

8. **For not one-one:**

$$1.1, 1.2 \in R(\text{domain})$$

now,  $1.1 \neq 1.2$  but  $f(1.1) = f(1.2) = 1 \Rightarrow f$  is not one-one.

**For not onto:**

Let  $\frac{1}{2} \in R$  (co-domain), but  $[x] = \frac{1}{2}$  is not possible for x in domain.

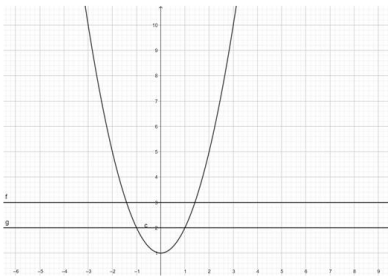
so, f is not onto.

9. To show:  $f: R \rightarrow R : f(x) = 1 + x^2$  is many - one into.

Proof:

$$f(x) = 1 + x^2$$

$$\text{or, } y = 1 + x^2$$



Since the lines cut the curve in 2 equal valued points of y, therefore, the function  $f(x)$  is many-one.

The range of  $f(x) = [1, \infty) \neq R$  (Codomain)

$\therefore f(x)$  is not onto

$\Rightarrow f(x)$  is into

Hence,  $f: R \rightarrow R : f(x) = 1 + x^2$  is many-one into.

10. f is one-one since each element of A is assigned to distinct element of the set A. Also, f is onto since  $f(A) = A$ . Moreover,  $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$ .

11. We have  $S = \{(a, b) : a \leq b^3\}$  where  $a, b \in R$ .

i. Reflexive: we observe that,  $\frac{1}{2} \leq \left(\frac{1}{2}\right)^3$  is not true.

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin S$ . So, S is not reflexive.

ii. Symmetric: We observe that  $1 \leq 3^3$  but  $3 \not\leq 1^3$  i.e.,  $(1, 3) \in S$  but  $(3, 1) \notin S$ . So, S is not symmetric.

iii. Transitive: We observe that,  $10 \leq 3^3$  and  $3 \leq 2^3$  but  $10 \not\leq 2^3$ .

i.e.,  $(10, 3) \in S$  and  $(3, 2) \in S$  but  $(10, 2) \notin S$ .

So, S is not transitive.

$\therefore$  S is neither reflexive nor symmetric, not transitive.

12. Given relation R on N defined by  $\{(a,b) : a \text{ is multiple of } b\}$ .

Here R satisfies the following properties:

1. Reflexivity

Let  $a$  be an arbitrary element of  $N$ . Then,  $a = (a \times 1)$

i.e  $a$  is a multiple of  $a$

$$\therefore (a, a) \in R \forall a \in N$$

So,  $R$  is reflexive.

2. Transitivity

let  $a, b, c \in N$  such that  $(a, b) \in R$  and  $(b, c) \in R$

$\Rightarrow a$  is a multiple of  $b$  and  $b$  is a multiple of  $c$

$\Rightarrow a = bd$  and  $b = ec$  for some  $d, e \in N$

$$\Rightarrow a = (ce)d$$

$$\Rightarrow a = c(ed)$$

$\Rightarrow a$  is a multiple of  $c$

$$\therefore (a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R$$

Hence,  $R$  is transitive.

3. Symmetric

Clearly, 6 and 2 are natural numbers and 6 is a multiple of 2.

$$\therefore (6, 2) \in R$$

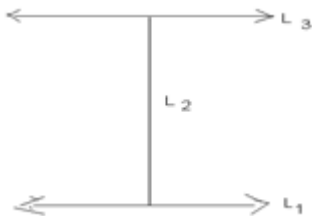
But 2 is not a multiple of 6.

$$\therefore (2, 6) \notin R$$

Thus,  $(6, 2) \in R$  and  $(2, 6) \notin R$

Hence,  $R$  is not symmetric.

13.  $R$  is not reflexive, as a line  $L_1$  cannot be  $\perp$  to itself i.e  $(L_1, L_1) \notin R$



Let  $(L_1, L_2) \in R$

$\Rightarrow L_1 \perp L_2$

$\Rightarrow L_2 \perp L_1$

$\Rightarrow (L_2, L_1) \in R$

$\Rightarrow R$  is symmetric

Let  $(L_1, L_2) \in R$  and  $(L_2, L_3) \in R$ , then

$L_1 \perp L_2$  and  $L_2 \perp L_3$

Then  $L_1$  can never be  $\perp$  to  $L_3$  in fact  $L_1 \parallel L_3$

i.e  $(L_1, L_2) \in R, (L_2, L_3) \in R$ .

But  $(L_1, L_3) \notin R$

$R$  is not transitive.

14. Given,  $f(x) = \frac{1}{x}$

For one-one Let  $x_1, x_2 \in R$ , such that  $f(x_1) = f(x_2)$

$$\Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \text{ [put } x_1 \text{ and } x_2 \text{ in } f(x) = \frac{1}{x} \text{ ]}$$

$$\Rightarrow x_1 = x_2$$

So,  $f$  is one-one

For onto Let  $y \in R$  be any arbitrary element.

Then,  $y = f(x)$

$$\Rightarrow y = \frac{1}{x}$$

$$\Rightarrow x = \frac{1}{y} \text{ [expressing } x \text{ in terms of } y \text{ ]}$$

It is clear that for every  $y \in R$  (codomain),  $x \in R$  (domain)

Thus, for each  $y \in R$  (codomain), there exist

$$x = \frac{1}{y} \in R \text{ (domain), such that } f(x) = f\left(\frac{1}{y}\right) = \frac{1}{\left(\frac{1}{y}\right)} = y$$

[i.e., every element of codomain has pre-image in domain]

So,  $f$  is onto.

15. Given a relation  $R$  in  $A \times A$  where  $A = \{1, 2, 3, \dots, 9\}$ , defined as  $(a, b) R (c, d)$ , if  $a + d = b + c$ .

**Reflexive:** Let  $(a, b)$  be any arbitrary element of  $A \times A$ ,

i.e.  $(a, b) \in A \times A$ , where  $a, b \in A$ .

Now, as  $a + b = b + a$  [addition is commutative]

$\therefore (a, b) R (a, b)$

So,  $R$  is reflexive.

**Symmetric:**

Let  $(a, b), (c, d) \in A \times A$  such that  $(a, b) R (c, d)$

Then,  $a + d = b + c$

$\Rightarrow b + c = a + d \Rightarrow c + b = d + a$  [addition is commutative]

$\Rightarrow (c, d) R (a, b)$

**Transitive:** Let  $(a, b), (c, d), (e, f) \in A \times A$  such that  $(a, b) R (c, d)$  and  $(c, d) R (e, f)$ .

Then,  $a + d = b + c$  and  $c + f = d + e$

On adding the above equations, we get

$a + d + c + f = b + c + d + e$

$\Rightarrow a + f = b + e \Rightarrow (a, b) R (e, f)$

So,  $R$  is transitive.

Thus,  $R$  is reflexive, symmetric and transitive.

Hence,  $R$  is an equivalence relation.

Now, for  $[(2, 5)]$ , we will find  $(c, d) \in A \times A$  such that  $2 + d = 5 + c$  or  $d - c = 3$

Clearly,  $(2, 5) R (1, 4)$  as  $4 - 1 = 3$

$(2, 5) R (2, 5)$  as  $5 - 2 = 3$

$(2, 5) R (3, 6)$  as  $6 - 3 = 3$

$(2, 5) R (4, 7)$  as  $7 - 4 = 3$

$(2, 5) R (5, 8)$  as  $8 - 5 = 3$

and  $(2, 5) R (6, 9)$  as  $9 - 6 = 3$

Hence, equivalence class  $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ .

16.  $f$  is one-one: For any  $x, y \in \mathbb{R}$ , we have  $f(x) : f(y)$

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{|y|+1}$$

$$\Rightarrow xy + x = xy + y$$

$$\Rightarrow x = y$$

Therefore,  $f$  is one-one function.

If  $f$  is one-one, let  $y \in \mathbb{R} - \{-1\}$ , then  $f(x) = y$

$$\Rightarrow \frac{x}{x+1} = y$$

$$\Rightarrow x = \frac{y}{1-y}$$

It is clear that  $x \in \mathbb{R}$  for all  $y \in \mathbb{R} - \{-1\}$ , also  $x \neq -1$

Because  $x = -1$

$$\Rightarrow \frac{y}{1-y} = -1$$

$\Rightarrow y = -1 + y$  which is not possible.

Thus for each  $\mathbb{R} - \{-1\}$  there exists  $x = \frac{y}{1-y} \in \mathbb{R} - \{-1\}$  such that

$$f(x) = \frac{x}{x+1} = \frac{\frac{y}{1-y}}{\frac{y}{1-y}+1} = y$$

Therefore  $f$  is onto function.

17. Given that  $A = \{2, 3, 4\}$ ,  $B = \{2, 5, 6, 7\}$

1. Let  $f : A \rightarrow B$  defined by

$$f = \{(x, y) : y = x + 3\}$$

i.e.  $f = \{(2, 5), (3, 6), (4, 7)\}$  which is an injective mapping.

2. Let  $g : A \rightarrow B$  denote a mapping such that  $g = \{(2, 2), (3, 5), (4, 5)\}$  which is not an injective mapping.

3. Let  $h : B \rightarrow A$  denote a mapping such that  $h = \{(2, 2), (5, 3), (6, 4), (7, 4)\}$  which is a mapping from  $B$  to  $A$ .

18.  $A = \mathbb{R} - \{3\}$ ,  $B = \mathbb{R} - \{1\}$

$f: A \rightarrow B$  is defined as  $f(x) = \left(\frac{x-2}{x-3}\right)$ .

Let  $x, y \in A$  such that  $f(x) = f(y)$ .

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$$

$$\Rightarrow -3x - 2y = -3y - 2x$$

$$\Rightarrow 3x - 2x = 3y - 2y$$

$$\Rightarrow x = y$$

Therefore,  $f$  is one-one.

Let  $y \in B = \mathbb{R} - \{1\}$

Then,  $y \neq 1$ .

The function  $f$  is onto if there exists  $x \in A$  such that  $f(x) = y$ .

Now,  $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x(1 - y) = -3y + 2$$

$$\Rightarrow x = \frac{2-3y}{1-y} \in \mathbf{A} \quad [y \neq 1]$$

Thus, for any  $y \in B$ , there exists  $\frac{2-3y}{1-y} \in \mathbf{A}$  such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right)-2}{\left(\frac{2-3y}{1-y}\right)-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$  is onto.

Hence, function  $f$  is one-one and onto.

19. i. Let  $f: \mathbb{N} \rightarrow \mathbb{N}$ , be a mapping defined by  $f(x) = 2x$

Which is one-one

For  $f(x_1) = f(x_2)$

$$\Rightarrow 2x_1 = 2x_2$$

$$x_1 = x_2$$

Further  $f$  is not onto, as for  $1 \in \mathbb{N}$ , there does not exist any  $x$  in  $\mathbb{N}$  such that  $f(x) = 1$ .

ii. Let  $f: \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(1) = f(2) = 1$  and  $f(x) = x - 1$  for every  $x > 2$  is onto but not one-one.  $f$  is not one-one as  $f(1) = f(2) = 1$ . But  $f$  is onto.

iii. The mapping  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined as  $f(x) = x^2$ , is neither one-one nor onto.