Matrices and Determinants - Matrix Operations and Exponentiation

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Question 15: Determinant of a Given Matrix

- The determinant is:

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 \end{vmatrix}$$

- Expanding using identities $a^x + a^{-x} = 2\cosh(x \ln a)$, the determinant simplifies.
- The final determinant evaluates to $2abc$.
- **Correct answer: (b)** $2abc$.

Question 16: Trigonometric Determinant Condition

- The determinant involves:

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos(nx) & \cos(n+1)x \\ \sin(n-1)x & \sin(nx) & \sin(n+1)x \end{vmatrix}$$

- The determinant is zero when either:
  1. $\sin x = 0$
  2. $\cos x = 0$
- Solving, we get $\cos x = \frac{1+a^2}{2a}$.
- **Correct answer: (d)** $\cos x = \frac{1+a^2}{2a}$.

Question 17: Determinant with Factorial Expressions

- The determinant involves factorial patterns:

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

- Using determinant properties and expansions, the final result simplifies.
- **Correct answer: (c)** $(n!)^3(2n^3 + 8n^2 + 10n + 4)$.

Question 18: Sum of Series Properties

- The determinant Δ involving sequences $S_k = a^k + b^k + c^k$ simplifies.

- By determinant properties, it reduces to $S_6 - S_4$.
- **Correct answer: (c)** $S_6 - S_4$.

Question 19: Non-Trivial Solutions for Two Polynomial Systems

- Given:

$$\alpha x + \alpha z = 0, \quad \beta y + \beta z = 0$$

- A non-trivial solution exists if:

$$\frac{b^2}{q^2} = \frac{c^2}{r^2} = \frac{ab}{pq}$$

- **Correct answer: (c)** $\frac{c^2}{r^2} = \frac{ab}{pq}$.

Question 20: Polynomial Division

- Given:

$$f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$$

- Polynomial division is performed to find g.
- The remainder theorem provides:
  - **Correct answer: (d)** -108.

Question 21: Matrix Invertibility and Properties

- Given:

$$AB = AC \neq 0$$

- Matrices do not necessarily cancel unless A is invertible.
- The valid statement is:
  - **Correct answer: (b) If $A^3 + 2A^2 + 3A + 5I = 0$, then A is invertible.**

Question 22: Finding Matrix Elements

- The determinant of:

$$\begin{vmatrix} a & 5x & p \\ b & 10y & 5 \\ c & 15z & 15 \end{vmatrix} = 125$$

- Solving gives:
  - **Correct answer: (a) 25**.

Question 23: Homogeneous System of Linear Equations

- The given equations:

$$-ax + y + z = 0$$

$$x - by + z = 0$$

$$x + y - cz = 0$$

- For a non-trivial solution, the determinant condition applies:

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$$

- **Correct answer: (b) 1**.

Question 24: Triangle Properties Using Determinants

- The determinant:

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0$$

- This equation suggests specific **triangle properties**.
- The triangle is **isosceles**.
- **Correct answer: (c) $\triangle ABC$ is an isosceles triangle**.

Key Concepts Covered:

1. **Factorization of Determinants**: Using matrix operations to simplify large expressions.
2. **Trigonometric Identities in Determinants**: Finding values that nullify a determinant.
3. **Factorial-Based Determinants**: Understanding recurrence in determinant solutions.
4. **Matrix Invertibility**: How determinant conditions impact solutions.
5. **Triangle Classification Using Determinants**: Conditions that identify triangle properties.
6. **Homogeneous System Solutions**: Applying determinant conditions for non-trivial solutions.

These concepts are fundamental in **linear algebra, number theory, and geometry**, with applications in **physics, engineering, and advanced mathematics**.