Solution

INTEGRALS SIMPLE AND SUBSTITUTION METHODS

Class 12 - Mathematics

1. $\int \frac{dx}{9+4x^2} = \frac{1}{6}\tan^{-1}\frac{2x}{3} + c$

2. Writing Integral as I = $\int \frac{x^3-x+x+1}{x^3-x}dx$

$= \int 1dx + \int \frac{1}{x(x-1)}dx$

$= x + \int \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$

= x + log|x - 1| - log |x| + c

3. Let (1 + x^2) = t

so 2x dx = dt

$\Rightarrow$ I = $\frac{1}{2}\int e'd = \frac{1}{2}e^t + c = \frac{1}{2}e^{(1+2^2)} + c$

4. $\int e^{2\log x}dx = \int x^2\,dx + c$

$= \frac{x^3}{3}$ + c

5. Let I = $\int \frac{\sin x}{\sin^3 x+\cos^3 x}dx$

$= \int \frac{\tan x\sec^2 x}{\tan^3 x+1}dx$

On substituting tan x = t and $\sec^2$x dx = dt, we get

$I = \int \frac{t}{t^3+1}dt$

$= \int \frac{t}{(t+1)(t^2-t+1)}dt$

$= -\frac{1}{3}\int \frac{1}{t+1}dt + \frac{1}{3}\int \frac{t+1}{t^2-t+1}dt$

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}dt + \frac{1}{2}\int \frac{1}{t^2-t+1}dt$

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$

$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)$

$= -\frac{1}{3}\log|\tan x+1| + \frac{1}{6}\log|\tan^2 x - \tan x + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right) + C$

6. Let the given integral be,

$I = \int \frac{2x}{(2x+1)^2}dx$

Now using partial fractions by putting, $\frac{2x}{(2x+1)^2} = \frac{A}{(2x+1)} + \frac{B}{(2x+1)^2}$ .... (1)

A(2x + 1) + B = 2x

Putting 2x + 1 = 0,

$x = \frac{-1}{2}$

A(0) + B = -1

B = -1

By equating the coefficient of x,

2A = 2

A = 1

From equation (1), we get,

$\frac{2x}{(2x+1)^2} = \frac{1}{(2x+1)} - \frac{1}{(2x+1)^2}$

$\int \frac{2x}{(2x+1)^2}dx = \int \frac{1}{(2x+1)}dx - \int \frac{1}{(2x+1)^2}dx$

$= \frac{\log|2x+1|}{2} + \frac{1}{2(2x+1)} + c$

$= \frac{1}{2}\left[\log|2x+1| + \frac{1}{2x+1}\right] + c$

7. Let I = $\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}dx$. Then,by rationalizing we have

$I = \int \frac{x\{\sqrt{x+a}-\sqrt{x+b}\}}{\{\sqrt{x+a}+\sqrt{x+b}\}\{\sqrt{x+a}-\sqrt{x+b}\}}dx = \int \frac{x\{\sqrt{x+a}-\sqrt{x+b}\}}{a-b}dx$

$\Rightarrow \quad I=\frac{1}{a-b} \int\{x \sqrt{x+a}-x \sqrt{x+b}\} d x$

$\Rightarrow \quad I=\frac{1}{a-b} \int\{(x+a-a) \sqrt{x+a}-(x+b-b) \sqrt{x+b}\} d x$

$\Rightarrow \quad I=\frac{1}{a-b} \int\left\{(x+a)^{3/2}-a \sqrt{x+a}-(x+b)^{3/2}+b \sqrt{x+b}\right\} d x$

$\Rightarrow \quad I=\frac{1}{a-b}\left\{\frac{2}{5}(x+a)^{5/2}-\frac{2a}{3}(x+a)^{3/2}-\frac{2}{5}(x+b)^{5/2}+\frac{2b}{3}(x+b)^{3/2}\right\}+C$

8. It is given that f '(x) = x + b

$\therefore \int$f '(x) = $\int$ (x + b) dx

$\Rightarrow$ f (x) = $\frac{x^2}{2}$ + bx + c ...(i)

Since,

f(1) = 5

$\therefore \frac{1^2}{2}$ + b $\times$ 1 + c = 5

$\Rightarrow \frac{1}{2}$ + b + c = 5

$\Rightarrow$ b + c = $\frac{9}{2}$ ...(ii)

and, f (2) = 13

$\Rightarrow \frac{(2)^2}{2}$ + b $\times$ 2 + c = 13

$\Rightarrow$ 2 + 2b + c = 13

$\Rightarrow$ 2b + c = 11 .....(iii)

Subtracting equation (ii) from equation (iii), we get

b = $11-\frac{9}{2}$

$\Rightarrow$ b = $\frac{13}{2}$

Putting b = $\frac{13}{2}$ in equation (ii), we get

$\frac{13}{2}$ + c = $\frac{9}{2}$

$\Rightarrow$ c = $\frac{9}{2}-\frac{13}{2}$

$\Rightarrow$ c = $\frac{9-13}{2}=\frac{-4}{2}$ = -2

Putting b = $\frac{13}{2}$ and c = -2 in equation (i), we get

f (x) = $\frac{x^2}{2}+\frac{13}{2}$x - 2

9. We have,

f(x) = $\int f'(x) dx$

$\Rightarrow$ f(x) = $\int(a \sin x+b \cos x) d x$

= -a cos x + b sin x + c

$\therefore$ f(x) = - a cos x + b sin x + c ...(i)

Since

f'(0) = 4

$\therefore$ f'(0) = a sin 0 + b cos 0 = 4

$\Rightarrow$ a $\times$ 0 + b = 4

$\Rightarrow$ b = 4

Now,

f(0) = 3

$\therefore$ f(0) = -a cos 0 + b sin 0 + c = 3

$\Rightarrow$ -a + 0 + c = 3

$\Rightarrow$ c - a = 3 ...(ii)

and, f$\left(\frac{\pi}{2}\right)$ = 5

$\therefore$ f$\left(\frac{\pi}{2}\right)$ = - a cos $\left(\frac{\pi}{2}\right)$ + b sin $\left(\frac{\pi}{2}\right)$ + c = 5

$\Rightarrow$ - a $\times$ 0 + b $\times$ 1 + c = 5

$\Rightarrow$ b + c = 5

$\Rightarrow$ 4 + c = 5 [$\because$ b = 4]

$\Rightarrow$ c = 5 - 4

$\Rightarrow$ c = 1

Putting c = 1 in equation (ii), we get

1 - a = 3

$\Rightarrow$ - a = 3 - 1

$\Rightarrow$ - a = 2

$\Rightarrow$ a = - 2

Putting a = - 2, b = 4 and c = 1 in equation (i), we get

f(x) = -(-2) cos x + 4 sin x + 1

$\Rightarrow$ f(x) = 2 cos x + 4 sin x + 1

Hence, f(x) = 2 cos x + 4 sin x + 1

10. let $\tan^{-1} x = t$

$\Rightarrow \frac{1}{1+x^2}dx = dt$

$\Rightarrow \int \frac{e^{tan^{-1}x}}{1+x^2}dx = \int e^t dt$

= e^t + c

= $e^{\tan^{-1}x} + C$

11. $\int cosecx dx = \int \frac{cosecx(\ cosecx - \cot x)}{(\ cosecx - \cot x)}dx$ [multiplying numerator and denominator by (cosec x - cot x)]

$= \int \frac{1}{t}dt$,where (cosec x - cot x) =t

= log | t | + C = log | cosec x - cot x | + C. [(cosec x - cot x) =t]

$\therefore \int cosecx dx = \log|\ cosecx - \cot x| + c$

12. Take sin x = t

So we get

cos x dx = dt

By integrating w.r.t. t

$\int t dt = \frac{t^2}{2} + c$

By substituting the value of t

$= \frac{(\sin x)^2}{2} + c$

13. Let I = $\int \tan^3 x \sec^2 x dx$

Put tan x = t

$\Rightarrow \sec^2 x dx = dt$

$\therefore\ I = \int t^3 dt$

$= \frac{t^4}{4}$+C

Hence, $I = \frac{\tan^4 x}{4} + c$

14. I = $\int \frac{1+\sin x}{\sqrt{x-\cos x}}dx$

put x - cos x = t

$\Rightarrow$ [1- (-sin x)]dx = dt

$\Rightarrow$ (1 + sin x) dx = dt

=$\int \frac{dt}{\sqrt{t}}$

=$\int t^{\frac{-1}{2}} dt$

=$2t^{\frac{1}{2}} + c$

=$2(x - \cos x)^{\frac{1}{2}} + c$

15. Rationalize the given integrand we get

$\Rightarrow \int \frac{\mathrm{x}}{\sqrt{\mathrm{x+a}} - \sqrt{\mathrm{x-b}}} \times \frac{\sqrt{\mathrm{x+a}} + \sqrt{\mathrm{x-b}}}{\sqrt{\mathrm{x+a}} + \sqrt{\mathrm{x-b}}}\mathrm{dx}$

$\Rightarrow \int \frac{x(\sqrt{x+a} - \sqrt{x-b})}{x+a-x-b}dx$

$\Rightarrow \int \frac{x(\sqrt{x+a} - \sqrt{x-b})}{a-b}dx$

$\Rightarrow \frac{1}{a-b}\int x(\sqrt{x+a} - \sqrt{x-b})dx$

Assume x = $\sqrt{t}$

$\Rightarrow \mathrm{dx} = \frac{\mathrm{dt}}{2\sqrt{\mathrm{t}}}$

Substituting values of t, and dt,

$\Rightarrow \int \sqrt{t}\frac{(\sqrt{\sqrt{t}+a} - \sqrt{\sqrt{t}-b})}{2\sqrt{t}(a-b)}dt$

$\Rightarrow \frac{1}{2(a-b)}\int(\sqrt{\sqrt{t}+a} - \sqrt{\sqrt{t}-b})dt$

$\Rightarrow \frac{1}{2(a-b)}\int(\sqrt{t}+a)^{1/2}dt - \int(\sqrt{t}-b)^{1/2}dt$

$\Rightarrow \frac{1}{2(a-b)}\left(\frac{4}{3}\left(\sqrt{t}+a^2\right)^{\frac{3}{2}}-\frac{4}{3}\left(t-a^2\right)^{\frac{3}{2}}\right)$

now replacing x = $\sqrt{t}$

$\Rightarrow \frac{1}{2(a-b)}\left(\frac{2}{3}(x+a)^{\frac{3}{2}}-\frac{2}{3}(x-b)^{\frac{3}{2}}\right)$

16. Let I = $\int \frac{1}{\sin x(3+2\cos x)} dx$

Multiply and divide numerator by sin x.

= $\int \frac{\sin x dx}{\sin^2 x(3+2\cos x)}$

= $\int \frac{\sin x dx}{(1-\cos^2 x)(3+2\cos x)}$

Let cos x = t

$\Rightarrow$ - sinxdx = dt

$\because$ I = $\int \frac{dt}{(t^2-1)(3+2t)}$

Now,

Let $\frac{1}{(t^2-1)(3+2t)}=\frac{A}{t-1}+\frac{B}{t+1}+\frac{C}{3+2t}$

$\Rightarrow$ 1 = A(t + 1) (3 + 2t) + B (t - 1) (3 + 2t) + C (t^2 - 1)

Put t = 1

$\Rightarrow$1 = 10A $\Rightarrow$ A = $\frac{1}{10}$

Put t = -1

$\Rightarrow$ 1 = -2B $\Rightarrow$ B = - $\frac{1}{2}$

Put t = -$\frac{3}{2}$

$\Rightarrow$ 1 = $\frac{5}{4}$, C $\Rightarrow$ C = $\frac{4}{5}$

Thus,

I = $\frac{1}{10}\int \frac{dt}{t-1}-\frac{1}{2}\int \frac{dt}{t+1}+\frac{5}{4}\int \frac{dt}{3+2t}$

= $\frac{1}{10}\log|t-1|-\frac{1}{2}\log|t+1|+\frac{2}{5}\log|3+2t|+c$

Hence,

I = $\frac{1}{10}\log|\cos x-1|-\frac{1}{2}\log|\cos x+1|+\frac{2}{5}\log|3+2\cos x|+c$.

17. Let the given integral be,

I = $\int \frac{1}{p+q\tan x} dx$

$=\int \frac{1}{p+q\left(\frac{\sin x}{\cos x}\right)} dx$

$=\int \frac{\cos x}{p\cos x+q\sin x} dx$

Let cos x = $\lambda\frac{d}{dx}$ (p cos x + q sin x)+ μ(p cos x + q sin x)+v

cos x = λ(-p sin x + q cos x)+μ(p cos x+q sin x) + v

cos x=(-p λ + q μ) sin x+(q λ + p μ) cos x + v

Comparing the coefficients of sin x, cos x on the both the sides,

-p λ + q μ = 0 ....(1)

q λ + p μ = 1 ....(2)

v = 0 ....(3)

Solving equation (1), (2) and (3), we get

$x=\frac{q}{(p^2+q^2)}$

$\mu=\frac{p}{(p^2+q^2)}$

v = 0

Now,

$I=\int \frac{q}{(p^2+q^2)}\frac{(-p\sin x+q\cos x)}{(p\cos x+q\sin x)}dx+\int \frac{p}{(p^2+q^2)}\frac{(p\cos x+q\sin x)}{(p\cos x+q\sin x)}dx$

$I=\frac{q}{(p^2+q^2)}(\log|p\cos x+q\sin x|)+\frac{p}{(p^2+q^2)}x+c$

18. According to the question , $I=\int \frac{\sqrt{x^2+1}\left[\log|x^2+1|-2\log|x|\right]}{x^4}dx$

$=\int \frac{\sqrt{x^2+1}\left[\log|x^2+1|-\log|(x)^2|\right]}{x^4}dx\ [\because mloga=loga^m]$

$= \int \frac{\sqrt{x^2+1}\log\left|\frac{x^2+1}{x^2}\right|}{x^4} dx \left[\because \log m - \log n = \log \frac{m}{n}\right]$

$= \int \frac{x\sqrt{1+\frac{1}{x^2}}\log\left|1+\frac{1}{x^2}\right|}{x^4} dx$

$= \int \frac{\sqrt{1+\frac{1}{x^2}}\log\left|1+\frac{1}{x^2}\right|}{x^3} dx$

Put , $1 + \frac{1}{x^2} = t$

$\Rightarrow \frac{-2}{x^3} dx = dt$

$\Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$

$\therefore \quad I = -\frac{1}{2}\int \sqrt{t}\log|t| dt$

By using integration by parts ,we get

$= -\frac{1}{2}\left[\log|t| \int t^{1/2} dt - \int \left\{\frac{d}{dt}(\log|t|) \int t^{1/2} dt\right\} dt\right]$

$= -\frac{1}{2}\left[\log|t| \times \frac{t^{3/2}}{3/2} - \int \frac{t^{3/2}}{3/2} \times \frac{1}{t} dt\right]$

$= -\frac{1}{3}\left[t^{3/2}\log|t| - \int \sqrt{t} dt\right]$

$= -\frac{1}{3}\left[t^{3/2}\log|t| - \frac{t^{3/2}}{3/2}\right] + C$

$= -\frac{1}{3}t^{3/2}\left[\log|t| - \frac{2}{3}\right] + C$

$= -\frac{1}{3}\left(1 + \frac{1}{x^2}\right)^{3/2}\left[\log\left|1 + \frac{1}{x^2}\right| - \frac{2}{3}\right] + C \left[\because t = 1 + \frac{1}{x^2}\right]$

19. suppose I = $\int \frac{x+\sqrt{x+1}}{x+2} dx$ ..(i)

Let x + 1 = t^2 then we have,

d(x + 1) = d(t^2)

=> dx = 2t dt

using x + 1 = t^2 and dx = 2t dt.. In equation (i), we have

$I = \int \frac{x+\sqrt{t^2}}{x+2} 2t dt$

$= 2\int \frac{(t^2-1)+t}{(t^2-1)+2} t dt \ since, x = t^2 - 1$

$= 2\int \frac{t^2+t-1}{t^2+1} t dt$

$= 2\int \frac{t^3+t^2-t}{t^2+1} dt$

$= 2\left[\int \frac{t^3}{t^2+1} dt + \int \frac{t^2}{t^2+1} dt - \int \frac{t}{t^2+1} dt\right]$

$\therefore \quad I = 2\left[\int \frac{t^3}{t^2+1} dt + \int \frac{t^2}{t^2+1} dt - 1\frac{t}{t^2+1} dt\right]$ ...(ii)

Let $I_1 = \int \frac{t^3}{t^2+1} dt$

$I_2 = \int \frac{t^2}{t^2+1} dt$

and $I_3 = \int \frac{t}{t^2+1} dt$

Now, $I_1 = \int \frac{t^3}{t^2+1} dt$

$= \int \left(t - \frac{t}{t^2+1}\right) dt$

$= \frac{t^2}{2} - \frac{1}{2}\log\left(t^2 + 1\right)$

$\therefore \quad I = \frac{t^2}{2} - \frac{1}{2}$ log(t^2 + 1) + c_1 ...(iii)

Since, $I_2 = \int \frac{t^2}{t^2+1} dt$

$= \int \frac{t^2+1-1}{t^2+1} dt$

$= \int \frac{t^2+1}{t^2+1} dt - \int \frac{1}{t^2+1} dt$

$= \int dt - \int \frac{1}{t^2+1} dt$

$\Rightarrow$ I_2 = t - $\tan^{-1}(t^2)$ + c_2 ....(iv)

and, $I_3 = \int \frac{t}{t^2+1} dt$

$= \frac{1}{2}$ log (1 + t^2) + c_3 ...(v)

Using equation (ii), (iii), (iv) and (v), we get

I = 2[$\frac{t^2}{2} - \frac{1}{2}$ log(t^2 + 1) + c_1 + t - $\tan^{-1}$(t^2) + c_2 - $\frac{1}{2}$log(1 + t^2) + c_3]

= 2 [$\frac{t^2}{2}$ + t - $\tan^{-1}$ (t^2) - log (1 + t^2) + c_1 + c_1 + c_3]

= 2 [$\frac{t^2}{2}$ + t - $\tan^{-1}$ (t^2) - log (1 + t^2) + c_4] [Putting c_1 + c_1 + c_3 = c_4]

= t + 2t - 2 $\tan^{-1}$ (t^2) - 2 log (1 + t^2) + 2c_4

= (x + 1) + $2\sqrt{x+1}$ - 2 $\tan^{-1}$($\sqrt{x+1}$) - 2 log(1 + x + 1) + 2c_4

= (x + 1) + $2\sqrt{x+1}$ - 2 $\tan^{-1}$($\sqrt{x+1}$) - 2 log(x + 2) + c [Putting 2c_4 = c]

$\therefore$ I = (x + 1) + $2\sqrt{x+1}$ - 2 $\tan^{-1}$($\sqrt{x+1}$) - 2 log(x + 2) + c