Partial Fractions Decomposition

Partial fractions

Partial fraction decomposition is a technique used to break down complex rational expressions into simpler fractions that are easier to integrate or differentiate. This method is particularly useful in calculus, especially when dealing with integral calculus.

Steps for Partial Fraction Decomposition:

1. **Factor the Denominator:**
   Ensure that the denominator of the rational function is factored completely. For example, if you have $\frac{P(x)}{Q(x)}$, make sure $Q(x)$ is factored into linear and/or irreducible quadratic factors.
2. **Set Up Partial Fractions:**
   Write the expression as a sum of fractions with unknown coefficients. The form depends on the factors of the denominator:
   - For linear factors: $\frac{A}{(x-a)}$
   - For repeated linear factors: $\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \cdots + \frac{N}{(x-a)^k}$
   - For irreducible quadratic factors: $\frac{Ax+B}{(x^2+bx+c)}$
   - For repeated irreducible quadratic factors: $\frac{Ax+B}{(x^2+bx+c)} + \frac{Cx+D}{(x^2+bx+c)^2} + \cdots + \frac{Mx+N}{(x^2+bx+c)^k}$
3. **Equate and Solve for Coefficients:**
   Multiply both sides by the common denominator to clear the fractions. Then, equate the coefficients of corresponding powers of x on both sides of the equation to form a system of linear equations. Solve this system to find the values of the unknown coefficients.
4. **Write the Decomposed Form:**
   Substitute the values of the coefficients back into the partial fractions.

Example Problem:

Decompose $\frac{2x+3}{(x-1)(x+2)}$.

1. **Factor the Denominator:**
   The denominator is already factored as $(x-1)(x+2)$.
2. **Set Up Partial Fractions:**

$$\frac{2x+3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

3. **Multiply and Equate:**
   Multiply both sides by $(x-1)(x+2)$:

$$2x + 3 = A(x+2) + B(x-1)$$

Expand and equate coefficients:

$$2x + 3 = Ax + 2A + Bx - B$$

Group like terms:

$$2x + 3 = (A+B)x + (2A - B)$$

Equate coefficients:

$$\begin{cases} A + B = 2 \\ 2A - B = 3 \end{cases}$$

4. **Solve the System:**
   From $A + B = 2$, we get $B = 2 - A$.
   Substitute B in the second equation:

$$\begin{gathered} 2A - (2 - A) = 3 \\ 2A - 2 + A = 3 \\ 3A = 5 \\ A = \frac{5}{3} \end{gathered}$$

   Then, $B = 2 - \frac{5}{3} = \frac{6}{3} - \frac{5}{3} = \frac{1}{3}$.
5. **Write the Decomposed Form:**

$$\frac{2x+3}{(x-1)(x+2)} = \frac{\frac{5}{3}}{x-1} + \frac{\frac{1}{3}}{x+2}$$