Integrals

Lecture Notes on Introduction to Integral Calculus

1. Motivation for Studying Integrals

Integral Calculus is a major part of mathematical analysis that focuses on the accumulation of quantities and the areas under and between curves. The motivation for studying integrals arises from various practical and theoretical problems, such as:

- **Determining Functions from Derivatives:** Given the derivative of a function, determining the original function (antiderivative) is a common problem. This process is known as finding the indefinite integral of the function.
- **Calculating Areas:** Another fundamental problem is calculating the area under a curve, which leads to the concept of definite integrals.
- **Applications in Physics:** In physics, integrals are used to determine quantities like displacement when velocity is known, or to find the total accumulated change over time.

2. Concept of Area Under a Curve

One of the primary motivations for the development of integral calculus is the need to calculate the area under curves. This concept can be broken down into several key points:

- **Definite Integrals:** The definite integral of a function over an interval $[a, b]$ provides the area under the curve of the function from $x = a$ to $x = b$.
- **Approximation Techniques:** Initially, the area under a curve can be approximated using methods like the Riemann sum, which involves dividing the area into small rectangles and summing their areas.
- **Limit Process:** As the number of rectangles increases and their width decreases, the approximation becomes more accurate, leading to the concept of the integral as a limit of these sums.

3. The Relationship Between Derivatives and Integrals

The Fundamental Theorem of Calculus establishes a profound connection between derivatives and integrals, which is divided into two parts:

- **First Part of the Fundamental Theorem of Calculus:** If F is an antiderivative of f on an interval $[a, b]$, then:

  $$\int_a^b f(x)\,dx = F(b) - F(a)$$

  This theorem links the concept of the definite integral to the antiderivative of a function, showing that the area under the curve $f(x)$ from a to b can be found using the antiderivative F.

- **Second Part of the Fundamental Theorem of Calculus:** If f is a continuous real-valued function on $[a, b]$ and F is defined by:

  $$F(x) = \int_a^x f(t)\, dt$$

  Then F is continuous on $[a, b]$, differentiable on (a, b), and $F'(x) = f(x)$. This shows that differentiation and integration are inverse processes.

Summary

Integral calculus is driven by the need to solve problems related to finding functions from their rates of change and calculating areas under curves. The development of integrals addresses these needs and connects deeply with the concept of derivatives through the Fundamental Theorem of Calculus. This theorem not only bridges the two areas of calculus but also provides a powerful tool for solving a wide range of problems in mathematics, physics, and engineering.