Min-Max Derivatives Overview

Lecture Notes on Application of Derivatives in Minima and Maxima

1. Introduction:
The application of derivatives in finding minima and maxima is a fundamental concept in calculus. It is used to determine the highest and lowest points on a graph, which has practical applications in various fields such as physics, engineering, economics, and more.
2. Critical Points:
To find the minima and maxima of a function, we first find its critical points.

- **Definition**: A critical point of a function $f(x)$ is a point where the derivative $f'(x) = 0$ or is undefined.
- **Procedure**:
    1. Compute the first derivative of the function $f(x)$.
    2. Set the derivative equal to zero and solve for x.
    3. Find points where the derivative does not exist (if any).

Example 1:
Find the critical points of $f(x) = x^3 - 3x^2 + 4$.
Solution:
$f'(x) = 3x^2 - 6x$
Set $f'(x) = 0$:
$3x^2 - 6x = 0$
$3x(x-2) = 0$
$x = 0 \quad \text{or} \quad x = 2$
So, the critical points are $x = 0$ and $x = 2$.
3. Second Derivative Test:
To determine whether each critical point is a local minimum, maximum, or neither, we use the second derivative test.

- **Procedure**:
    1. Compute the second derivative $f''(x)$.
    2. Evaluate $f''(x)$ at each critical point.
    3. Determine the nature of each critical point:
        - If $f''(x) > 0$, the function has a local minimum at x.
        - If $f''(x) < 0$, the function has a local maximum at x.
        - If $f''(x) = 0$, the test is inconclusive.

Example 2:
Determine the nature of the critical points of $f(x) = x^3 - 3x^2 + 4$.
Solution:
$f''(x) = 6x - 6$

Evaluate the second derivative at the critical points:

- At $x = 0$:
  $f''(0) = 6(0) - 6 = -6$
  Since $f''(0) < 0$, $x = 0$ is a local maximum.
- At $x = 2$:
  $f''(2) = 6(2) - 6 = 6$
  Since $f''(2) > 0$, $x = 2$ is a local minimum.

4. Global (Absolute) Minima and Maxima:
To find the global minima and maxima on a closed interval $[a, b]$, evaluate the function at critical points and at the endpoints of the interval.

- **Procedure**:
  1. Find all critical points within the interval $[a, b]$.
  2. Evaluate the function at each critical point and at the endpoints a and b.
  3. Compare these values to determine the global minimum and maximum.

Example 3:
Find the global minimum and maximum of $f(x) = x^3 - 3x^2 + 4$ on the interval $[-1, 3]$.
Solution:
Critical points are $x = 0$ and $x = 2$.
Evaluate $f(x)$ at the critical points and endpoints:
$f(-1) = (-1)^3 - 3(-1)^2 + 4 = -1 - 3 + 4 = 0$
$f(0) = (0)^3 - 3(0)^2 + 4 = 4$
$f(2) = (2)^3 - 3(2)^2 + 4 = 8 - 12 + 4 = 0$
$f(3) = (3)^3 - 3(3)^2 + 4 = 27 - 27 + 4 = 4$
The global minimum is 0 (occurs at $x = -1$ and $x = 2$), and the global maximum is 4 (occurs at $x = 0$ and $x = 3$).
5. Practical Applications:

- **Economics**: Finding the maximum profit or minimum cost.
- **Physics**: Determining the points of maximum or minimum potential energy.
- **Engineering**: Optimizing design parameters for maximum efficiency or strength.

Example 4:
A company produces and sells x units of a product. The revenue and cost functions are given by $R(x) = 50x$ and $C(x) = 30x + 200$. Find the number of units x that maximize the profit.
Solution:
Profit function $P(x) = R(x) - C(x)$:
$P(x) = 50x - (30x + 200) = 20x - 200$
First derivative:
$P'(x) = 20$
Since $P'(x) \neq 0$, there are no critical points within the domain.
Evaluate at the endpoints of the practical domain, typically starting from 0 units up to a reasonable maximum capacity. If no specific interval is given, assume the production can be any non-negative number:

The profit increases linearly with x, hence producing more units always increases profit up to the capacity limit.
Summary:

1. Find the first derivative to identify critical points.
2. Use the second derivative test to classify these points.
3. Evaluate the function at critical points and endpoints to find global extrema.

Use these steps to solve practical problems in various fields.