Solution

COMPLEX NUMBERS

Class 11 - Mathematics

1. We have,
   (a + b) - i(3a + 2b) = 5 + 2i
   ⇒ a + b = 5 and -(3a + 2b) = 2 [comparing real and imaginary parts]
   ⇒ a = -12, b = 17
2. We have,
   $z_1 = z_2$ ⇒ 2 - iy = x + 3i
   ⇒ 2 = x and - y = 3 [comparing real and imaginary part]
   ⇒ x = 2 and y = -3
3. (2 + i3) + (- 6 + i7) = (2 - 6) + i (3 + 7) = - 4 + i10
4. Given, $(5+4i)+(5-4i)$
   = (5 + 5) + i(4 - 4) = 10 + 0i
5. (1 - i) - (-1 + i6)
   1 - i + 1 - 6i = 2 - 7i
6. (- 4 + 7i) - (- 11 - 23i) = (- 4 + 7i) + (11 + 23i)
   = (- 4 + 11) + (7 + 23) i
   = 7 + 30 i
7. Reciprocal of z = $\frac{\bar{z}}{|z|^2}$
   Therefore, reciprocal of 3 + $\sqrt{7}$ i= $\frac{3-\sqrt{7}i}{16}=\frac{3}{16}-\frac{\sqrt{7}i}{16}$
8. $\frac{1}{3-4i}=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}$
   $=\frac{3+4i}{9-16i^2}=\frac{3+4i}{9+16}$
   $\frac{1}{3-4i}$ = $\frac{3}{25}+\frac{4}{25}i.$
9. (x + iy) (4 + 5i) = 6 - 2i
   $\Rightarrow 4x+5xi+4yi+5yi^2=6-2i$
   $\Rightarrow (4x-5y)+(5x+4y)i=6-2i$
   Comparing real and imaginary parts on both sides, we have
   4x - 5y = 6 and 5x + 4y = - 2
   Solving these two equations for x and y , we get
   $x=\frac{14}{41}$ and $y=-\frac{38}{41}$
10. $\left[\frac{1}{1-4i}-\frac{2}{1+i}\right]\left[\frac{3-4i}{5+i}\right]=\left[\frac{1+i-2+8i}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$
    $=\left[\frac{-1+9i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right]=\left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$
    $=\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}=\frac{33+31i}{28-10i}\times\frac{28+10i}{28+10i}$
    $=\frac{924+330i+868i+310i^2}{(28)^2-(10i)^2}=\frac{614+1198i}{784+100}(\because i^2=-1)$
    $=\frac{2(307+599i)}{884}=\frac{307+599i}{442}$
11. $\frac{5+4i}{4+5i}=\frac{5+4i}{4+5i}\times\frac{4-5i}{4-5i}=\frac{(20+20)+i(16-25)}{16-25i^2}$ = $\frac{40-9i}{41}=\frac{40}{41}-\frac{9}{41}i$
12. $\frac{2+3i}{4+5i}$
    $=\frac{2+3i}{4+5i}\times\frac{4-5i}{4-5i}$ [multiply and divide by 4-5i]
    $=\frac{8-10i+12i-15i^2}{16-25i^2}$ ($\because$ i^2 = -1)
    $=\frac{23+2i}{16+25}$
    $=\frac{23}{41}+\frac{2}{41}i$
13. n = 2, because $(1 + i)^{2n} = (1 – i)^{2n}$ = $\left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^{2n}$ = 1
    $\Rightarrow \left(\frac{1+\ 2i\ +\ i^2}{1-\ i^2}\right)^{2n}$ = 1

$\Rightarrow \left(\frac{1+\ 2i\ -1}{1+1}\right)^{2n}$

$\Rightarrow$ (i)^{2n} = 1 which is possible if n = 2 ($\because$ i^4 = 1)

14. $(-2i)\left(\frac{1}{6}i\right) = \left(-2 \times \frac{1}{6}\right) \times i^2$

$= \frac{-1}{3} \times (-1) = \frac{1}{3}$

=> $(-2i)\left(\frac{1}{6}i\right)$ = $\frac{1}{3}$

15. Let, (a + ib)^2 = 0 + 4i

$\Rightarrow$ a^2 + (bi)^2 + 2abi = 0 + 4i [(a + b)^2 = a^2 + b^2 + 2ab]

$\Rightarrow$ a^2 - b^2 + 2abi = 0 + 4i [i^2 = -1]

Now, separating real and complex parts, we get

$\Rightarrow$ a^2 - b^2 = 0 …………..(1)

$\Rightarrow$ 2ab = 4

$\Rightarrow$ a = $\frac{2}{b}$ …….. (2)

Now, using the value of a in (1), we get

$\left(\frac{2}{b}\right)^2$ - b^2 = 0

$\Rightarrow$4 - b^4 = 0

$\Rightarrow$ b^4 = 4

Simplify and get the value of b^2 , we get,

$\Rightarrow$ b^2 = -2 or b^2 = 2

As b is real no. so, b^2 = 2

b= $\sqrt{2}$ or b= $-\sqrt{2}$

put value of b in equation (2) ==> a= $\sqrt{2}$ or a= $-\sqrt{2}$

Hence the square root of the complex no. is $\sqrt{2}+\sqrt{2}i$ and $-\sqrt{2}-\sqrt{2}i$.

16. Let the square root of i be x + iy

$\Rightarrow$ $\sqrt{i}$ = x + iy

$\Rightarrow$ i = x^2 + y^{2}i^2 + 2ixy (Squaring both the sides)

$\Rightarrow$ i = x^2 - y^2 + 2ixy

Comparing both the sides:

x^2 - y^2 = 0 ... (i)

and 2xy = 1 ... (ii)

By equation (ii), we find that x and y are of the same sign.

From equation (i)

x^2 = y^2

$\Rightarrow x = \pm y$

$\therefore xy = \frac{1}{2}, x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$

$\therefore \sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$

17. Let x+ yi = $\sqrt{1}$

Squaring both sides, we get

(x + yi)^2 = i

x^2 - y^2 + 2xyi = i

Equating the real and imaginary parts

x^2 - y^2 = 0........ (i)

$2\text{xy} = 1 \Rightarrow \text{xy} = \frac{1}{2}$

From the identity

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$(x^2+y^2)^2$

$= (0)^2 + 4\left(\frac{1}{2}\right)^2$

= 1

$\therefore$ $x^2 + y^2 = 1$ .... (ii) [Neglecting (-) sign as $x^2 + y^2 > 0$]

Solving (i) and (ii) we get

$x^2 = \frac{1}{2}$ and $y^2 = \frac{1}{2}$

$\therefore x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{\sqrt{2}}$

Since the sign of xy is positive then if $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$

If $x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2}}$

$\therefore \sqrt{1} = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

18. We have, $x - iy = \sqrt{\frac{a-ib}{c-id}}$

On squaring both sides, we get

$x^2 - y^2 - 2ixy = \frac{a-ib}{c-id}$ [$\because i^2 = -1$]

$= \frac{a-ib}{c-id} \times \frac{c+id}{c+id}$ [multiplying numerator and denominator by c + id]

$= \frac{ac-(i)^2(bd)-ibc+iad}{c^2+d^2}$

$= \frac{ac+bd-ibc+iad}{c^2+d^2}$

$\therefore x^2 - y^2 - 2ixy = \frac{ac+bd}{c^2+d^2} + i\left(\frac{ad-bc}{c^2+d^2}\right)$

On equating real and imaginary parts both sides, we get

$x^2 - y^2 = \frac{ac+bd}{c^2+d^2}$ and $2xy = \frac{bc-ad}{c^2+d^2}$

Now, we know that,

$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$

$= \left[\frac{ac+bd}{c^2+d^2}\right]^2 + \left[\frac{bc-ad}{c^2+d^2}\right]^2$

$= \frac{a^2c^2+b^2d^2+2abcd+b^2c^2+a^2d^2-2abcd}{\left(c^2+d^2\right)^2}$

$= \frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{\left(c^2+d^2\right)^2}$

$= \frac{\left(c^2+d^2\right)\left(a^2+b^2\right)}{\left(c^2+d^2\right)^2} = \frac{a^2+b^2}{c^2+d^2}$

$\therefore (x^2 + y^2) = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

Hence proved.

19. We have $\left|\frac{z-2}{z-3}\right| = 2$

putting = x + iy, we get

$\left|\frac{x+iy-2}{x+iy-3}\right| = 2$

$\frac{|x+iy-2|}{|x+iy-3|} = 2$

$\Rightarrow |x - 2 + iy| = 2|x - 3 + iy| \Rightarrow \sqrt{(x-2)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$

Squaring both the sides, we get

$\Rightarrow x^2 - 4x + 4 + y^2 = 4(x^2 - 6x + 9 + y^2) \Rightarrow 3x^2 + 3y^2 - 20x + 32 = 0$

$\Rightarrow x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0 \Rightarrow \left(x - \frac{10}{3}\right)^2 + y^2 + \frac{32}{3} - \frac{100}{9} = 0$

$\Rightarrow \left(x - \frac{10}{3}\right)^2 + (y - 0)2 = \frac{4}{9}$

Hence, the centre of the circle is $\left(\frac{10}{3}, 0\right)$ and radius is $\frac{2}{3}$.

20. $z_1 = r_1 \cos\theta_1 + i\sin\theta_1$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1| = r_1, |z_2| = r_2$, arg $(z_1) = \theta_1$ and arg $(z_2) = \theta_2$

i. We have,

$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore |z_1 + z_2|^2 = (r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$\Rightarrow |z_1 + z_2|$ = $r_1^2 + r_2^2 + 2r_1r_2 \cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

ii. We have,

$z_1 - z_2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i(r_1 \sin\theta_1 - r_2 \sin\theta_2)$

$\therefore |z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$

$\Rightarrow |z_1 - z_2|$ = $r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)$

$\Rightarrow |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

21. Let z = x + iy

Now, we have, $\left|\frac{z-5i}{z+5i}\right| = 1 \Rightarrow \frac{|z-5i|}{|z+5i|} = 1 \left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$

$\Rightarrow |z - 5i| = |z + 5i|$

$\Rightarrow |x + (y - 5)i|^2 = |x + (y + 5)i|^2$ [$\because$ put z = x + iy]

$\Rightarrow x^2 + (y - 5)^2 = x^2 + (y + 5)^2$

$\Rightarrow y^2 + 25 - 10y = y^2 + 25 + 10y$

$\Rightarrow 20y = 0$

$\Rightarrow y = 0$

$\therefore z = x + iy = x + i\,0 = x$

which represents that he is not sensitive towards environment. Hence, the person is not Eco-friendly. He needs to be Eco-friendly by taking remedial measures.

22. Let z = x + iy

Given: $|z| = z + 1 + 2i$

$\Rightarrow |x + iy| = x + iy + 1 + 2i$

$\Rightarrow \sqrt{x^2 + y^2} = (x + 1) + i(y + 2)$

$\Rightarrow x^2 + y^2 = (x + 1)^2 + 2i(x + 1)(y + 2) - (y + 2)^2$ [Squaring both sides]

$\Rightarrow x^2 + y^2 = x^2 + 2x + 1 + 2i(xy + 2x + y + 2) - (y^2 + 4y + 4)$

$\Rightarrow 2y^2 - 2x + 4y + 3 = 2i(xy + 2x + y + 2)$

$\Rightarrow y^2 - x + 2y + 2 = i(xy + 2x + y + 2)$

$\Rightarrow (y^2 - x + 2y + 2) - i(xy + 2x + y + 2) = 0$

On comparing we get,

$(xy + 2x + y + 2) = 0$

$\Rightarrow (x + 1)(y + 2) = 0$

$\Rightarrow x = -1, y = -2$

Also, $(y^2 - x + 2y + 2) = 0$

Taking x = - 1, $(y^2 - (-1) + 2y + 2) = 0$

$\Rightarrow (y^2 + 2y + 3) = 0$

Does not have a solution since roots will be imaginary.

Taking y = −2, $(4 - x - 4 + 2) = 0$

$\Rightarrow x = 2$

$\therefore z = x + iy = 2 - 2i$